

Problem 16

In each of Problems 13 through 16, find the solution of the given initial value problem. Plot the graph of the solution and describe how the solution behaves as $x \rightarrow 0$.

$$x^2 y'' + 3xy' + 5y = 0, \quad y(1) = 1, \quad y'(1) = -1$$

Solution

This is a homogeneous equidimensional (Euler) ODE, so the solution is of the form $y = x^r$.

$$y = x^r \quad \rightarrow \quad y' = rx^{r-1} \quad \rightarrow \quad y'' = r(r-1)x^{r-2}$$

Substitute these expressions into the ODE.

$$x^2 r(r-1)x^{r-2} + 3rx^{r-1} + 5x^r = 0$$

$$r(r-1)x^r + 3rx^r + 5x^r = 0$$

Divide both sides by x^r .

$$r(r-1) + 3r + 5 = 0$$

Solve for r .

$$r^2 + 2r + 5 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$r = \{-1 - 2i, -1 + 2i\}$$

Two solutions to the ODE are $y = x^{-1-2i}$ and $y = x^{-1+2i}$. According to the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} y(x) &= C_1 x^{-1-2i} + C_2 x^{-1+2i} \\ &= C_1 x^{-1} x^{-2i} + C_2 x^{-1} x^{2i} \\ &= C_1 x^{-1} e^{\ln x^{-2i}} + C_2 x^{-1} e^{\ln x^{2i}} \\ &= C_1 x^{-1} e^{-2i \ln x} + C_2 x^{-1} e^{2i \ln x} \\ &= C_1 x^{-1} [\cos(-2 \ln x) + i \sin(-2 \ln x)] + C_2 x^{-1} [\cos(2 \ln x) + i \sin(2 \ln x)] \\ &= C_1 x^{-1} [\cos(2 \ln x) - i \sin(2 \ln x)] + C_2 x^{-1} [\cos(2 \ln x) + i \sin(2 \ln x)] \\ &= (C_1 + C_2) x^{-1} \cos(2 \ln x) + (-iC_1 + iC_2) x^{-1} \sin(2 \ln x) \\ &= C_3 x^{-1} \cos(2 \ln x) + C_4 x^{-1} \sin(2 \ln x) \end{aligned}$$

Differentiate it with respect to x .

$$y'(x) = -C_3 x^{-2} \cos(2 \ln x) - C_3 x^{-1} \sin(2 \ln x) \cdot \frac{2}{x} - C_4 x^{-2} \sin(2 \ln x) + C_4 x^{-1} \cos(2 \ln x) \cdot \frac{2}{x}$$

Now apply the initial conditions, $y(1) = 1$ and $y'(1) = -1$, to determine C_3 and C_4 .

$$y(1) = C_3 = 1$$

$$y'(1) = -C_3 + 2C_4 = -1$$

Solving this system of equations yields $C_3 = 1$ and $C_4 = 0$. Therefore,

$$y(x) = x^{-1} \cos(2 \ln x).$$

The limit of $y(x)$ as $x \rightarrow 0$ is undefined.

Below is a plot of $y(x)$ versus x .

