Problem 20

In each of Problems 17 through 34, find all singular points of the given equation and determine whether each one is regular or irregular.

$$x^2(1-x^2)y'' + (2/x)y' + 4y = 0$$

Solution

The coefficient of y'' has zeros at $x = 0$ and $x = 1$ and $x = -1$, which means $x = 0$ and $x = 1$ and $x = -1$ are singular points. To determine whether they are regular or irregular, divide both sides of the ODE by $x^2(1-x^2)$

$$y'' + \frac{2}{x^3(1-x^2)}y' + \frac{4}{x^2(1-x^2)}y = 0$$

and compute the following limits.

$$\lim_{x\to 0} x\frac{2}{x^3(1-x^2)} = \lim_{x\to 0}\frac{2}{x^2(1-x^2)} = \infty$$

$$\lim_{x\to 0} x^2\frac{2}{x^3(1-x^2)} = \lim_{x\to 0}\frac{2}{x(1-x^2)} = \infty$$

$$\lim_{x\to 1}(x-1)\frac{2}{x^3(1-x^2)} = \lim_{x\to 1}(1-x)\frac{-2}{x^3(1-x)(1+x)} = \lim_{x\to 1}\frac{-2}{x^3(1+x)} = -1$$

$$\lim_{x\to 1}(x-1)^2\frac{4}{x^2(1-x^2)} = \lim_{x\to 1}(1-x)^2\frac{4}{x^2(1-x)(1+x)} = \lim_{x\to 1}\frac{4(1-x)}{x^2(1+x)} = 0$$

$$\lim_{x\to -1}(x+1)\frac{2}{x^3(1-x^2)} = \lim_{x\to -1}(1+x)\frac{2}{x^3(1-x)(1+x)} = \lim_{x\to -1}\frac{2}{x^3(1-x)} = -1$$

$$\lim_{x\to -1}(x+1)^2\frac{4}{x^2(1-x^2)} = \lim_{x\to -1}(1+x)^2\frac{4}{x^2(1-x)(1+x)} = \lim_{x\to -1}\frac{4(1+x)}{x^2(1-x)} = 0$$

Because at least one of the limits as $x \to 0$ is infinite, $x = 0$ is an irregular singular point. However, because both limits as $x \to 1$ and $x \to -1$ are finite, $x = 1$ and $x = -1$ are regular singular points.