Problem 21

In each of Problems 17 through 34, find all singular points of the given equation and determine whether each one is regular or irregular.

$$(1-x^2)^2y'' + x(1-x)y' + (1+x)y = 0$$

Solution

The coefficient of y'' has zeros at $x = 1$ and $x = -1$, which means $x = 1$ and $x = -1$ are singular points. To determine whether they are regular or irregular, divide both sides of the ODE by $(1-x^2)^2$

$$y'' + \frac{x(1-x)}{(1-x^2)^2}y' + \frac{1+x}{(1-x^2)^2}y = 0$$

$$y'' + \frac{x(1-x)}{(1-x)^2(1+x)^2}y' + \frac{1+x}{(1-x)^2(1+x)^2}y = 0$$

$$y'' + \frac{x}{(1-x)(1+x)^2}y' + \frac{1}{(1-x)^2(1+x)}y = 0$$

and compute the following limits.

$$\lim_{x\to 1}(x-1)\frac{x}{(1-x)(1+x)^2} = \lim_{x\to 1}(1-x)\frac{-x}{(1-x)(1+x)^2} = \lim_{x\to 1}\frac{-x}{(1+x)^2} = -\frac{1}{4}$$

$$\lim_{x\to 1}(x-1)^2\frac{1}{(1-x)^2(1+x)} = \lim_{x\to 1}(1-x)^2\frac{1}{(1-x)^2(1+x)} = \lim_{x\to 1}\frac{1}{1+x} = \frac{1}{2}$$

$$\lim_{x\to -1}(x+1)\frac{x}{(1-x)(1+x)^2} = \lim_{x\to -1}(1+x)\frac{x}{(1-x)(1+x)^2} = \lim_{x\to -1}\frac{x}{(1-x)(1+x)} = \infty$$

$$\lim_{x\to -1}(x+1)^2\frac{1}{(1-x)^2(1+x)} = \lim_{x\to -1}(1+x)^2\frac{1}{(1-x)^2(1+x)} = \lim_{x\to -1}\frac{1+x}{(1-x)^2} = 0$$

Because both limits as $x \to 1$ are finite, $x = 1$ is a regular singular point. However, because at least one of the limits as $x \to -1$ is infinite, $x = -1$ is an irregular singular point.