Problem 26

In each of Problems 17 through 34, find all singular points of the given equation and determine whether each one is regular or irregular.

$$x(3-x)y'' + (x+1)y' - 2y = 0$$

Solution

The coefficient of y'' has zeros at $x = 0$ and $x = 3$, which means $x = 0$ and $x = 3$ are singular points. To determine whether they are regular or irregular, divide both sides of the ODE by $x(3-x)$

$$y'' + \frac{x+1}{x(3-x)}y' - \frac{2}{x(3-x)}y = 0$$

and compute the following limits.

$$\lim_{x\to 0} x\frac{x+1}{x(3-x)} = \lim_{x\to 0}\frac{x+1}{3-x} = \frac{1}{3}$$

$$\lim_{x\to 0} x^2\left[-\frac{2}{x(3-x)}\right] = \lim_{x\to 0}\frac{-2x}{3-x} = 0$$

$$\lim_{x\to 3}(x-3)\frac{x+1}{x(3-x)} = -\lim_{x\to 3}\frac{x+1}{x} = -\frac{4}{3}$$

$$\lim_{x\to 3}(x-3)^2\left[-\frac{2}{x(3-x)}\right] = \lim_{x\to 3}(3-x)^2\left[-\frac{2}{x(3-x)}\right] = \lim_{x\to 3}\frac{-2(3-x)}{x} = 0$$

Because both limits as $x \to 0$ and $x \to 3$ are finite, $x = 0$ and $x = 3$ are regular singular points.