Problem 5

Find the Laplace transform of each of the following functions:

(a) $f(t) = t$

(b) $f(t) = t^2$

(c) $f(t) = t^n$, where n is a positive integer

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Find the Laplace transform of t.

$$\begin{aligned}
\mathcal{L}\{t\} &= \int_0^\infty te^{-st}\, dt \\
&= \int_0^\infty \left(-\frac{\partial}{\partial s} e^{-st}\right) dt \\
&= -\frac{d}{ds}\int_0^\infty e^{-st}\, dt \\
&= -\frac{d}{ds}\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty \\
&= -\frac{d}{ds}\left(\frac{1}{s}\right) \qquad (1) \\
&= \frac{1}{s^2}
\end{aligned}$$

Find the Laplace transform of t^2.

$$\begin{aligned}
\mathcal{L}\{t^2\} &= \int_0^\infty t^2 e^{-st}\, dt \\
&= \int_0^\infty \left(\frac{\partial^2}{\partial s^2} e^{-st}\right) dt \\
&= \frac{d^2}{ds^2}\int_0^\infty e^{-st}\, dt \\
&= \frac{d^2}{ds^2}\left(-\frac{1}{s}e^{-st}\right)\Big|_0^\infty \\
&= \frac{d^2}{ds^2}\left(\frac{1}{s}\right) \qquad (2) \\
&= \frac{2}{s^3}
\end{aligned}$$

Note that for equations (1) and (2) to be true, it is critical that $s > 0$. Generalizing the results, we see that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$