Problem 6

Find the Laplace transform of $f(t) = \cos at$, where a is a real constant.

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Find the Laplace transform of $\cos at$.

$$\begin{aligned}
\mathcal{L}\{\cos at\} &= \int_0^\infty e^{-st} \cos at\, dt \\
&= \int_0^\infty e^{-st} \frac{e^{iat} + e^{-iat}}{2}\, dt \\
&= \frac{1}{2} \int_0^\infty e^{-st} (e^{iat} + e^{-iat})\, dt \\
&= \frac{1}{2} \left(\int_0^\infty e^{iat-st}\, dt + \int_0^\infty e^{-iat-st}\, dt \right) \\
&= \frac{1}{2} \left[\int_0^\infty e^{(ia-s)t}\, dt + \int_0^\infty e^{(-ia-s)t}\, dt \right] \\
&= \frac{1}{2} \left[\frac{1}{ia-s} e^{(ia-s)t} \bigg|_0^\infty + \frac{1}{-ia-s} e^{(-ia-s)t} \bigg|_0^\infty \right] \\
&= \frac{1}{2} \left[\frac{1}{ia-s}(-1) + \frac{1}{-ia-s}(-1) \right] \qquad (1) \\
&= \frac{1}{2} \left(\frac{1}{s-ia} + \frac{1}{s+ia} \right) \\
&= \frac{1}{2} \left[\frac{s+ia+s-ia}{(s-ia)(s+ia)} \right] \\
&= \frac{1}{2} \left(\frac{2s}{s^2 - i^2 a^2} \right) \\
&= \frac{s}{s^2 + a^2}
\end{aligned}$$

Note that for equation (1) to hold, it is critical that $s > 0$.