Problem 9

Recall that $\cosh bt = (e^{bt} + e^{-bt})/2$ and $\sinh bt = (e^{bt} - e^{-bt})/2$. In each of Problems 7 through 10, find the Laplace transform of the given function; a and b are real constants.

$$f(t) = e^{at}\cosh bt$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Find the Laplace transform of $e^{at}\cosh bt$.

$$\begin{aligned}
\mathcal{L}\{e^{at}\cosh bt\} &= \int_0^\infty e^{-st}e^{at}\cosh bt\, dt \\
&= \int_0^\infty e^{-st}e^{at}\frac{e^{bt}+e^{-bt}}{2}\, dt \\
&= \frac{1}{2}\int_0^\infty e^{-st}e^{at}(e^{bt}+e^{-bt})\, dt \\
&= \frac{1}{2}\left(\int_0^\infty e^{bt-st+at}\, dt + \int_0^\infty e^{-bt-st+at}\, dt\right) \\
&= \frac{1}{2}\left[\int_0^\infty e^{(a+b-s)t}\, dt + \int_0^\infty e^{(a-b-s)t}\, dt\right] \\
&= \frac{1}{2}\left[\frac{1}{a+b-s}e^{(a+b-s)t}\bigg|_0^\infty + \frac{1}{a-b-s}e^{(a-b-s)t}\bigg|_0^\infty\right] \\
&= \frac{1}{2}\left[\frac{1}{a+b-s}(-1) + \frac{1}{a-b-s}(-1)\right] \qquad (1) \\
&= \frac{1}{2}\left(\frac{1}{s-a-b} + \frac{1}{s-a+b}\right) \\
&= \frac{1}{2}\left[\frac{s-a+b+s-a-b}{(s-a-b)(s-a+b)}\right] \\
&= \frac{1}{2}\left(\frac{2s-2a}{s^2-2as+a^2-b^2}\right) \\
&= \frac{s-a}{(s-a)^2-b^2}
\end{aligned}$$

Note that for equation (1) to hold, it is critical that $a+b-s<0$ and $a-b-s<0$, that is,

$$s-a > b \quad \text{and} \quad s-a > -b$$

$$s-a > |b|.$$