

Problem 10

Recall that $\cosh bt = (e^{bt} + e^{-bt})/2$ and $\sinh bt = (e^{bt} - e^{-bt})/2$. In each of Problems 7 through 10, find the Laplace transform of the given function; a and b are real constants.

$$f(t) = e^{at} \sinh bt$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

Find the Laplace transform of $e^{at} \sinh bt$.

$$\begin{aligned} \mathcal{L}\{e^{at} \sinh bt\} &= \int_0^{\infty} e^{-st} e^{at} \sinh bt dt \\ &= \int_0^{\infty} e^{-st} e^{at} \frac{e^{bt} - e^{-bt}}{2} dt \\ &= \frac{1}{2} \int_0^{\infty} e^{-st} e^{at} (e^{bt} - e^{-bt}) dt \\ &= \frac{1}{2} \left(\int_0^{\infty} e^{bt-st+at} dt - \int_0^{\infty} e^{-bt-st+at} dt \right) \\ &= \frac{1}{2} \left[\int_0^{\infty} e^{(a+b-s)t} dt - \int_0^{\infty} e^{(a-b-s)t} dt \right] \\ &= \frac{1}{2} \left[\frac{1}{a+b-s} e^{(a+b-s)t} \Big|_0^{\infty} - \frac{1}{a-b-s} e^{(a-b-s)t} \Big|_0^{\infty} \right] \\ &= \frac{1}{2} \left[\frac{1}{a+b-s} (-1) - \frac{1}{a-b-s} (-1) \right] \\ &= \frac{1}{2} \left(\frac{1}{s-a-b} - \frac{1}{s-a+b} \right) \\ &= \frac{1}{2} \left[\frac{s-a+b - (s-a-b)}{(s-a-b)(s-a+b)} \right] \\ &= \frac{1}{2} \left(\frac{2b}{s^2 - 2as + a^2 - b^2} \right) \\ &= \frac{b}{(s-a)^2 - b^2} \end{aligned} \tag{1}$$

Note that for equation (1) to hold, it is critical that $a+b-s < 0$ and $a-b-s < 0$, that is,

$$s-a > b \quad \text{and} \quad s-a > -b$$

$$s-a > |b|.$$