

## Problem 11

Recall that  $\cos bt = (e^{ibt} + e^{-ibt})/2$  and that  $\sin bt = (e^{ibt} - e^{-ibt})/2i$ . In each of Problems 11 through 14, find the Laplace transform of the given function;  $a$  and  $b$  are real constants. Assume that the necessary elementary integration formulas extend to this case.

$$f(t) = \sin bt$$

### Solution

The Laplace transform of a function  $f(t)$  is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

Find the Laplace transform of  $\sin bt$ .

$$\begin{aligned} \mathcal{L}\{\sin bt\} &= \int_0^{\infty} e^{-st} \sin bt dt \\ &= \int_0^{\infty} e^{-st} \frac{e^{ibt} - e^{-ibt}}{2i} dt \\ &= \frac{1}{2i} \int_0^{\infty} e^{-st} (e^{ibt} - e^{-ibt}) dt \\ &= \frac{1}{2i} \left( \int_0^{\infty} e^{ibt-st} dt - \int_0^{\infty} e^{-ibt-st} dt \right) \\ &= \frac{1}{2i} \left[ \int_0^{\infty} e^{(ib-s)t} dt - \int_0^{\infty} e^{(-ib-s)t} dt \right] \\ &= \frac{1}{2i} \left[ \frac{1}{ib-s} e^{(ib-s)t} \Big|_0^{\infty} - \frac{1}{-ib-s} e^{(-ib-s)t} \Big|_0^{\infty} \right] \\ &= \frac{1}{2i} \left[ \frac{1}{ib-s} (-1) - \frac{1}{-ib-s} (-1) \right] \tag{1} \\ &= \frac{1}{2i} \left( \frac{1}{s-ib} - \frac{1}{s+ib} \right) \\ &= \frac{1}{2i} \left[ \frac{s+ib - (s-ib)}{(s-ib)(s+ib)} \right] \\ &= \frac{1}{2i} \left( \frac{2ib}{s^2 - i^2 b^2} \right) \\ &= \frac{b}{s^2 + b^2} \end{aligned}$$

Note that for equation (1) to hold, it is critical that  $s > 0$ .