Problem 16

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

$$f(t) = t\sin at$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Find the Laplace transform of $t\sin at$.

$$\begin{aligned}
\mathcal{L}\{t\sin at\} &= \int_0^\infty e^{-st} t\sin at\,dt \\
&= \int_0^\infty e^{-st} t\frac{e^{iat}-e^{-iat}}{2i}\,dt \\
&= \frac{1}{2i}\left(\int_0^\infty te^{iat-st}\,dt - \int_0^\infty te^{-iat-st}\,dt\right) \\
&= \frac{1}{2i}\left[\int_0^\infty te^{(ia-s)t}\,dt - \int_0^\infty te^{(-ia-s)t}\,dt\right] \\
&= \frac{1}{2i}\left\{\int_0^\infty t\frac{d}{dt}\left[\frac{1}{ia-s}e^{(ia-s)t}\right]dt - \int_0^\infty t\frac{d}{dt}\left[\frac{1}{-ia-s}e^{(-ia-s)t}\right]dt\right\} \\
&= \frac{1}{2i}\left\{t\left[\frac{1}{ia-s}e^{(ia-s)t}\right]\Bigg|_0^\infty - \int_0^\infty (1)\left[\frac{1}{ia-s}e^{(ia-s)t}\right]dt\right. \\
&\qquad\left. - t\left[\frac{1}{-ia-s}e^{(-ia-s)t}\right]\Bigg|_0^\infty + \int_0^\infty (1)\left[\frac{1}{-ia-s}e^{(-ia-s)t}\right]dt\right\} \\
&= \frac{1}{2i}\left[-\frac{1}{ia-s}\int_0^\infty e^{(ia-s)t}\,dt + \frac{1}{-ia-s}\int_0^\infty e^{(-ia-s)t}\,dt\right] \qquad (1) \\
&= \frac{1}{2i}\left[-\frac{1}{(ia-s)^2}e^{(ia-s)t}\Bigg|_0^\infty + \frac{1}{(-ia-s)^2}e^{(-ia-s)t}\Bigg|_0^\infty\right] \\
&= \frac{1}{2i}\left[-\frac{1}{(ia-s)^2}(-1) + \frac{1}{(-ia-s)^2}(-1)\right] \qquad (2) \\
&= \frac{1}{2i}\left[\frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2}\right] \\
&= \frac{1}{2i}\left[\frac{(s+ia)^2-(s-ia)^2}{(s-ia)^2(s+ia)^2}\right] \\
&= \frac{1}{2i}\left(\frac{4ias}{[(s-ia)(s+ia)]^2}\right) \\
&= \frac{2as}{(s^2+a^2)^2}
\end{aligned}$$

Note that for equations (1) and (2) to hold, it is critical that $s > 0$.