Problem 17

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

$$f(t) = t\cosh at$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Find the Laplace transform of $t\cosh at$.

$$\begin{aligned}
\mathcal{L}\{t\cosh at\} &= \int_0^\infty e^{-st} t\cosh at\,dt \\
&= \int_0^\infty e^{-st} t\frac{e^{at}+e^{-at}}{2}\,dt \\
&= \frac{1}{2}\left(\int_0^\infty te^{-st+at}\,dt + \int_0^\infty te^{-st-at}\,dt\right) \\
&= \frac{1}{2}\left[\int_0^\infty te^{(a-s)t}\,dt + \int_0^\infty te^{(-s-a)t}\,dt\right] \\
&= \frac{1}{2}\left\{\int_0^\infty t\frac{d}{dt}\left[\frac{1}{a-s}e^{(a-s)t}\right]dt + \int_0^\infty t\frac{d}{dt}\left[\frac{1}{-s-a}e^{(-s-a)t}\right]dt\right\} \\
&= \frac{1}{2}\left\{t\left[\frac{1}{a-s}e^{(a-s)t}\right]\Bigg|_0^\infty - \int_0^\infty (1)\left[\frac{1}{a-s}e^{(a-s)t}\right]dt\right. \\
&\qquad \left. + t\left[\frac{1}{-s-a}e^{(-s-a)t}\right]\Bigg|_0^\infty - \int_0^\infty (1)\left[\frac{1}{-s-a}e^{(-s-a)t}\right]dt\right\} \\
&= \frac{1}{2}\left[-\frac{1}{a-s}\int_0^\infty e^{(a-s)t}\,dt - \frac{1}{-s-a}\int_0^\infty e^{(-s-a)t}\,dt\right] && (1) \\
&= \frac{1}{2}\left[-\frac{1}{(a-s)^2}e^{(a-s)t}\Bigg|_0^\infty - \frac{1}{(-s-a)^2}e^{(-s-a)t}\Bigg|_0^\infty\right] \\
&= \frac{1}{2}\left[-\frac{1}{(a-s)^2}(-1) - \frac{1}{(-s-a)^2}(-1)\right] && (2) \\
&= \frac{1}{2}\left[\frac{1}{(a-s)^2} + \frac{1}{(-s-a)^2}\right] \\
&= \frac{1}{2}\left[\frac{1}{(s-a)^2} + \frac{1}{(s+a)^2}\right] \\
&= \frac{1}{2}\left[\frac{(s+a)^2+(s-a)^2}{(s-a)^2(s+a)^2}\right] \\
&= \frac{1}{2}\left[\frac{2s^2+2a^2}{[(s-a)(s+a)]^2}\right] = \frac{s^2+a^2}{(s^2-a^2)^2}
\end{aligned}$$

In order for equations (1) and (2) to hold, it's necessary that $a - s < 0$ and $-s - a < 0$, that is, $s > a$ and $s > -a$, or $s > |a|$.