

Problem 22

In each of Problems 21 through 24, find the Laplace transform of the given function.

$$f(t) = \begin{cases} t, & 0 \leq t < 1 \\ 0, & 1 \leq t < \infty \end{cases}$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

Split up the integral over the intervals that $f(t)$ is defined on.

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^1 e^{-st}(t) dt + \int_1^{\infty} e^{-st}(0) dt \\ &= \int_0^1 te^{-st} dt \\ &= \int_0^1 \left(-\frac{\partial}{\partial s} e^{-st}\right) dt \\ &= -\frac{d}{ds} \int_0^1 e^{-st} dt \\ &= -\frac{d}{ds} \left(-\frac{1}{s} e^{-st} \Big|_0^1\right) \\ &= -\frac{d}{ds} \left[\frac{1}{s}(1 - e^{-s})\right] \\ &= -\left[-\frac{1}{s^2}(1 - e^{-s}) + \frac{1}{s}e^{-s}\right] \\ &= -\left(\frac{-1 + e^{-s} + se^{-s}}{s^2}\right) \\ &= \frac{1 - e^{-s} - se^{-s}}{s^2} \\ &= \frac{1 - (1 + s)e^{-s}}{s^2} \end{aligned}$$

Below is a side-by-side comparison of the function $f(t)$ and its Laplace transform $F(s)$.

