Problem 30

The Gamma Function. The gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) = \int_0^\infty e^{-x} x^p \, dx. \tag{i}$$

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper at $x = 0$, because the integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.

(a) Show that, for $p > 0$,

$$\Gamma(p+1) = p\Gamma(p).$$

(b) Show that $\Gamma(1) = 1$.

(c) If p is a positive integer n, show that

$$\Gamma(n+1) = n!$$

Since $\Gamma(p)$ is also defined when p is not an integer, this function provides an extension of the factorial function to nonintegral values of the independent variable. Note that it is also consistent to define $0! = 1$.

(d) Show that, for $p > 0$,

$$p(p+1)(p+2)\cdots(p+n-1) = \Gamma(p+n)/\Gamma(p).$$

Thus $\Gamma(p)$ can be determined for all positive values of p if $\Gamma(p)$ is known in a single interval of unit length—say, $0 < p \leq 1$. It is possible to show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Find $\Gamma\left(\frac{3}{2}\right)$ and $\Gamma\left(\frac{11}{2}\right)$.

Solution

Part (a)

Use integration by parts in the definition of the gamma function.

$$\begin{aligned}
\Gamma(p+1) &= \int_0^\infty e^{-x} x^p \, dx \\
&= \int_0^\infty \frac{d}{dx}(-e^{-x}) x^p \, dx \\
&= (-e^{-x}) x^p \Big|_0^\infty - \int_0^\infty (-e^{-x}) p x^{p-1} \, dx \\
&= p \int_0^\infty e^{-x} x^{p-1} \, dx \\
&= p\Gamma(p)
\end{aligned}$$

Note that l'Hôpital's rule can be applied Ceiling(p) (that is, p rounded up to the nearest integer) times to show that the limit of $(-e^{-x})x^p$ as $x \to \infty$ is zero.

Part (b)

Set $p = 0$ to get $\Gamma(1)$ in the definition.

$$\begin{aligned}\Gamma(0+1) &= \int_0^\infty e^{-x} x^0 \, dx \\ \Gamma(1) &= \int_0^\infty e^{-x} \, dx \\ &= -e^{-x}\Big|_0^\infty \\ &= \lim_{x\to\infty} (-e^{-x}) - (-e^0) \\ &= 0 + 1 \\ &= 1\end{aligned}$$

Part (c)

Set $p = n$ in the result of part (a) and use it over and over again until $\Gamma(1)$ is reached. Then use the result of part (b).

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) \\ &= n[(n-1)\Gamma(n-1)] \\ &= n(n-1)\Gamma(n-1) \\ &= n(n-1)[(n-2)\Gamma(n-2)] \\ &= n(n-1)(n-2)\Gamma(n-2) \\ &= n(n-1)(n-2)[(n-3)\Gamma(n-3)] \\ &= n(n-1)(n-2)(n-3)\Gamma(n-3) \\ &\vdots \\ &= n(n-1)(n-2)(n-3)\cdots(3)(2)(1)\Gamma(1) \\ &= n(n-1)(n-2)(n-3)\cdots(3)(2)(1)(1) \\ &= n(n-1)(n-2)(n-3)\cdots(3)(2)(1) \\ &= n!\end{aligned}$$

Part (d)

Use the result of part (a) in the numerator over and over again until $\Gamma(p)$ is reached.

$$\begin{aligned}
\frac{\Gamma(p+n)}{\Gamma(p)} &= \frac{(p+n-1)\Gamma(p+n-1)}{\Gamma(p)} \\
&= \frac{(p+n-1)(p+n-2)\Gamma(p+n-2)}{\Gamma(p)} \\
&= \frac{(p+n-1)(p+n-2)\cdots(p+2)\Gamma(p+2)}{\Gamma(p)} \\
&= \frac{(p+n-1)(p+n-2)\cdots(p+2)(p+1)\Gamma(p+1)}{\Gamma(p)} \\
&= \frac{(p+n-1)(p+n-2)\cdots(p+2)(p+1)p\Gamma(p)}{\Gamma(p)} \\
&= (p+n-1)(p+n-2)\cdots(p+2)(p+1)p
\end{aligned}$$

Find $\Gamma\left(\frac{3}{2}\right)$.

$$\begin{aligned}
\frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}\right)} &= \left(\frac{1}{2}+1-1\right) \\
\frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}} &= \frac{1}{2} \\
\Gamma\left(\frac{3}{2}\right) &= \frac{\sqrt{\pi}}{2}
\end{aligned}$$

Find $\Gamma\left(\frac{11}{2}\right)$.

$$\begin{aligned}
\frac{\Gamma\left(\frac{1}{2}+5\right)}{\Gamma\left(\frac{1}{2}\right)} &= \left(\frac{1}{2}+5-1\right)\left(\frac{1}{2}+4-1\right)\left(\frac{1}{2}+3-1\right)\left(\frac{1}{2}+2-1\right)\left(\frac{1}{2}+1-1\right) \\
\frac{\Gamma\left(\frac{11}{2}\right)}{\sqrt{\pi}} &= \left(\frac{9}{2}\right)\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \\
\Gamma\left(\frac{11}{2}\right) &= \frac{945}{32}\sqrt{\pi}
\end{aligned}$$