

Problem 13

Recall that $\cos bt = (e^{ibt} + e^{-ibt})/2$ and that $\sin bt = (e^{ibt} - e^{-ibt})/2i$. In each of Problems 11 through 14, find the Laplace transform of the given function; a and b are real constants. Assume that the necessary elementary integration formulas extend to this case.

$$f(t) = e^{at} \sin bt$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

Find the Laplace transform of $e^{at} \sin bt$.

$$\begin{aligned} \mathcal{L}\{e^{at} \sin bt\} &= \int_0^{\infty} e^{-st} e^{at} \sin bt dt \\ &= \int_0^{\infty} e^{-st} e^{at} \frac{e^{ibt} - e^{-ibt}}{2i} dt \\ &= \frac{1}{2i} \int_0^{\infty} e^{-st} e^{at} (e^{ibt} - e^{-ibt}) dt \\ &= \frac{1}{2i} \left(\int_0^{\infty} e^{ibt-st+at} dt - \int_0^{\infty} e^{-ibt-st+at} dt \right) \\ &= \frac{1}{2i} \left[\int_0^{\infty} e^{(a+ib-s)t} dt - \int_0^{\infty} e^{(a-ib-s)t} dt \right] \\ &= \frac{1}{2i} \left[\frac{1}{a+ib-s} e^{(a+ib-s)t} \Big|_0^{\infty} - \frac{1}{a-ib-s} e^{(a-ib-s)t} \Big|_0^{\infty} \right] \\ &= \frac{1}{2i} \left[\frac{1}{a+ib-s} (-1) - \frac{1}{a-ib-s} (-1) \right] \tag{1} \\ &= \frac{1}{2i} \left(\frac{1}{s-a-ib} - \frac{1}{s-a+ib} \right) \\ &= \frac{1}{2i} \left[\frac{s-a+ib - (s-a-ib)}{(s-a-ib)(s-a+ib)} \right] \\ &= \frac{1}{2i} \left(\frac{2ib}{s^2 - 2as + a^2 + b^2} \right) \\ &= \frac{b}{(s-a)^2 + b^2} \end{aligned}$$

Note that for equation (1) to hold, it is critical that $a - s < 0$, that is,

$$s > a.$$