Problem 14

Recall that $\cos bt = (e^{ibt} + e^{-ibt})/2$ and that $\sin bt = (e^{ibt} - e^{-ibt})/2i$. In each of Problems 11 through 14, find the Laplace transform of the given function; a and b are real constants. Assume that the necessary elementary integration formulas extend to this case.

$$f(t) = e^{at}\cos bt$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Find the Laplace transform of $e^{at}\cos bt$.

$$\begin{aligned}
\mathcal{L}\{e^{at}\cos bt\} &= \int_0^\infty e^{-st}e^{at}\cos bt\, dt \\
&= \int_0^\infty e^{-st}e^{at}\frac{e^{ibt}+e^{-ibt}}{2}\, dt \\
&= \frac{1}{2}\int_0^\infty e^{-st}e^{at}(e^{ibt}+e^{-ibt})\, dt \\
&= \frac{1}{2}\left(\int_0^\infty e^{ibt-st+at}\, dt + \int_0^\infty e^{-ibt-st+at}\, dt\right) \\
&= \frac{1}{2}\left[\int_0^\infty e^{(a+ib-s)t}\, dt + \int_0^\infty e^{(a-ib-s)t}\, dt\right] \\
&= \frac{1}{2}\left[\frac{1}{a+ib-s}e^{(a+ib-s)t}\bigg|_0^\infty + \frac{1}{a-ib-s}e^{(a-ib-s)t}\bigg|_0^\infty\right] \\
&= \frac{1}{2}\left[\frac{1}{a+ib-s}(-1) + \frac{1}{a-ib-s}(-1)\right] \qquad (1) \\
&= \frac{1}{2}\left(\frac{1}{s-a-ib} + \frac{1}{s-a+ib}\right) \\
&= \frac{1}{2}\left[\frac{s-a+ib+(s-a-ib)}{(s-a-ib)(s-a+ib)}\right] \\
&= \frac{1}{2}\left(\frac{2s-2a}{s^2-2as+a^2+b^2}\right) \\
&= \frac{s-a}{(s-a)^2+b^2}
\end{aligned}$$

Note that for equation (1) to hold, it is critical that $a - s < 0$, that is,

$$s > a.$$