Problem 18

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

$$f(t) = t^n e^{at}$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Find the Laplace transform of $t^n e^{at}$. It's important to assume that $a - s < 0$, that is, $s > a$.

$$\begin{aligned}
\mathcal{L}\{t^n e^{at}\} &= \int_0^\infty e^{-st} t^n e^{at}\, dt \\
&= \int_0^\infty t^n e^{-st+at}\, dt \\
&= \int_0^\infty t^n e^{(a-s)t}\, dt \\
&= \int_0^\infty t^n \frac{d}{dt}\left[\frac{1}{a-s} e^{(a-s)t}\right] dt \\
&= t^n \left[\frac{1}{a-s} e^{(a-s)t}\right]\Bigg|_0^\infty - \int_0^\infty n t^{n-1} \left[\frac{1}{a-s} e^{(a-s)t}\right] dt \\
&= -\frac{n}{a-s} \int_0^\infty t^{n-1} e^{(a-s)t}\, dt \\
&= -\frac{n}{a-s} \int_0^\infty t^{n-1} \frac{d}{dt}\left[\frac{1}{a-s} e^{(a-s)t}\right] dt \\
&= -\frac{n}{a-s} \left\{ t^{n-1} \left[\frac{1}{a-s} e^{(a-s)t}\right]\Bigg|_0^\infty - \int_0^\infty (n-1) t^{n-2} \left[\frac{1}{a-s} e^{(a-s)t}\right] dt \right\} \\
&= -\frac{n}{a-s} \left[-\frac{n-1}{a-s} \int_0^\infty t^{n-2} e^{(a-s)t}\, dt\right] \\
&= \frac{n(n-1)}{(a-s)^2} \int_0^\infty t^{n-2} e^{(a-s)t}\, dt \\
&\vdots \\
&= \frac{n!}{(s-a)^n} \int_0^\infty e^{(a-s)t}\, dt \\
&= \frac{n!}{(s-a)^n} \frac{1}{a-s} e^{(a-s)t}\Bigg|_0^\infty \\
&= \frac{n!}{(s-a)^n} \frac{1}{a-s}(-1) \\
&= \frac{n!}{(s-a)^{n+1}}
\end{aligned}$$