Problem 19

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

$$f(t) = t^2 \sin at$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Find the Laplace transform of $t^2 \sin at$.

$$\begin{aligned}
\mathcal{L}\{t^2 \sin at\} &= \int_0^\infty e^{-st} t^2 \sin at\, dt \\
&= \int_0^\infty e^{-st} t^2 \frac{e^{iat} - e^{-iat}}{2i}\, dt \\
&= \frac{1}{2i}\left(\int_0^\infty t^2 e^{iat-st}\, dt - \int_0^\infty t^2 e^{-iat-st}\, dt\right) \\
&= \frac{1}{2i}\left[\int_0^\infty t^2 e^{(ia-s)t}\, dt - \int_0^\infty t^2 e^{(-ia-s)t}\, dt\right] \\
&= \frac{1}{2i}\left\{\int_0^\infty t^2 \frac{d}{dt}\left[\frac{1}{ia-s} e^{(ia-s)t}\right] dt - \int_0^\infty t^2 \frac{d}{dt}\left[\frac{1}{-ia-s} e^{(-ia-s)t}\right] dt\right\} \\
&= \frac{1}{2i}\left\{ t^2 \left[\frac{1}{ia-s} e^{(ia-s)t}\right]\Bigg|_0^\infty - \int_0^\infty (2t)\left[\frac{1}{ia-s} e^{(ia-s)t}\right] dt \right. \\
&\qquad \left. - t^2 \left[\frac{1}{-ia-s} e^{(-ia-s)t}\right]\Bigg|_0^\infty + \int_0^\infty (2t)\left[\frac{1}{-ia-s} e^{(-ia-s)t}\right] dt \right\} \\
&= \frac{1}{2i}\left[-\frac{2}{ia-s}\int_0^\infty t e^{(ia-s)t}\, dt + \frac{2}{-ia-s}\int_0^\infty t e^{(-ia-s)t}\, dt\right] \\
&= \frac{1}{2i}\left\{-\frac{2}{ia-s}\int_0^\infty t \frac{d}{dt}\left[\frac{1}{ia-s} e^{(ia-s)t}\right] dt + \frac{2}{-ia-s}\int_0^\infty t \frac{d}{dt}\left[\frac{1}{-ia-s} e^{(-ia-s)t}\right] dt\right\} \\
&= \frac{1}{2i}\left\{ -\frac{2}{ia-s}\left\{ t\left[\frac{1}{ia-s} e^{(ia-s)t}\right]\Bigg|_0^\infty - \int_0^\infty (1)\left[\frac{1}{ia-s} e^{(ia-s)t}\right] dt\right\}\right. \\
&\qquad \left. + \frac{2}{-ia-s}\left\{ t\left[\frac{1}{-ia-s} e^{(-ia-s)t}\right]\Bigg|_0^\infty - \int_0^\infty (1)\left[\frac{1}{-ia-s} e^{(-ia-s)t}\right] dt\right\}\right\} \\
&= \frac{1}{2i}\left\{-\frac{2}{ia-s}\left[-\frac{1}{ia-s}\int_0^\infty e^{(ia-s)t}\, dt\right] + \frac{2}{-ia-s}\left[-\frac{1}{-ia-s}\int_0^\infty e^{(-ia-s)t}\, dt\right]\right\} \\
&= \frac{1}{2i}\left[\frac{2}{(ia-s)^2}\int_0^\infty e^{(ia-s)t}\, dt - \frac{2}{(-ia-s)^2}\int_0^\infty e^{(-ia-s)t}\, dt\right] \\
&= \frac{1}{2i}\left[\frac{2}{(ia-s)^2}\frac{1}{(ia-s)} e^{(ia-s)t}\Bigg|_0^\infty - \frac{2}{(-ia-s)^2}\frac{1}{(-ia-s)} e^{(-ia-s)t}\Bigg|_0^\infty\right]
\end{aligned}$$

Assuming that $s > 0$, the exponential functions tend to zero as $t \to \infty$.

$$\begin{aligned}
\mathcal{L}\{t^2 \sin at\} &= \frac{1}{2i}\left[\frac{2}{(ia-s)^3}(-1) - \frac{2}{(-ia-s)^3}(-1)\right] \\
&= \frac{1}{i}\left[-\frac{1}{(ia-s)^3} + \frac{1}{(-ia-s)^3}\right] \\
&= \frac{1}{i}\left[\frac{1}{(s-ia)^3} - \frac{1}{(s+ia)^3}\right] \\
&= \frac{1}{i}\left[\frac{(s+ia)^3 - (s-ia)^3}{(s-ia)^3(s+ia)^3}\right] \\
&= \frac{1}{i}\left[\frac{6ias^2 - 2ia^3}{[(s-ia)(s+ia)]^3}\right] \\
&= \frac{6as^2 - 2a^3}{(s^2+a^2)^3} \\
&= 2a\frac{3s^2 - a^2}{(s^2+a^2)^3}
\end{aligned}$$