Problem 20

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

$$f(t) = t^2 \sinh at$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Find the Laplace transform of $t^2 \sinh at$.

$$\begin{aligned}
\mathcal{L}\{t^2 \sinh at\} &= \int_0^\infty e^{-st} t^2 \sinh at\, dt \\
&= \int_0^\infty e^{-st} t^2 \frac{e^{at} - e^{-at}}{2}\, dt \\
&= \frac{1}{2}\left(\int_0^\infty t^2 e^{at-st}\, dt - \int_0^\infty t^2 e^{-at-st}\, dt\right) \\
&= \frac{1}{2}\left[\int_0^\infty t^2 e^{(a-s)t}\, dt - \int_0^\infty t^2 e^{(-a-s)t}\, dt\right] \\
&= \frac{1}{2}\left\{\int_0^\infty t^2 \frac{d}{dt}\left[\frac{1}{a-s}e^{(a-s)t}\right] dt - \int_0^\infty t^2 \frac{d}{dt}\left[\frac{1}{-a-s}e^{(-a-s)t}\right] dt\right\} \\
&= \frac{1}{2}\Bigg\{ t^2 \left[\frac{1}{a-s}e^{(a-s)t}\right]\Bigg|_0^\infty - \int_0^\infty (2t)\left[\frac{1}{a-s}e^{(a-s)t}\right] dt \\
&\qquad - t^2 \left[\frac{1}{-a-s}e^{(-a-s)t}\right]\Bigg|_0^\infty + \int_0^\infty (2t)\left[\frac{1}{-a-s}e^{(-a-s)t}\right] dt \Bigg\} \\
&= \frac{1}{2}\left[-\frac{2}{a-s}\int_0^\infty te^{(a-s)t}\, dt + \frac{2}{-a-s}\int_0^\infty te^{(-a-s)t}\, dt\right] \\
&= \frac{1}{2}\left\{-\frac{2}{a-s}\int_0^\infty t\frac{d}{dt}\left[\frac{1}{a-s}e^{(a-s)t}\right] dt + \frac{2}{-a-s}\int_0^\infty t\frac{d}{dt}\left[\frac{1}{-a-s}e^{(-a-s)t}\right] dt\right\} \\
&= \frac{1}{2}\Bigg\{ -\frac{2}{a-s}\left\{t\left[\frac{1}{a-s}e^{(a-s)t}\right]\Bigg|_0^\infty - \int_0^\infty (1)\left[\frac{1}{a-s}e^{(a-s)t}\right] dt\right\} \\
&\qquad + \frac{2}{-a-s}\left\{t\left[\frac{1}{-a-s}e^{(-a-s)t}\right]\Bigg|_0^\infty - \int_0^\infty (1)\left[\frac{1}{-a-s}e^{(-a-s)t}\right] dt\right\}\Bigg\} \\
&= \frac{1}{2}\left\{-\frac{2}{a-s}\left[-\frac{1}{a-s}\int_0^\infty e^{(a-s)t}\, dt\right] + \frac{2}{-a-s}\left[-\frac{1}{-a-s}\int_0^\infty e^{(-a-s)t}\, dt\right]\right\} \\
&= \frac{1}{2}\left[\frac{2}{(a-s)^2}\int_0^\infty e^{(a-s)t}\, dt - \frac{2}{(-a-s)^2}\int_0^\infty e^{(-a-s)t}\, dt\right] \\
&= \frac{1}{2}\left[\frac{2}{(a-s)^2}\frac{1}{(a-s)}e^{(a-s)t}\Bigg|_0^\infty - \frac{2}{(-a-s)^2}\frac{1}{(-a-s)}e^{(-a-s)t}\Bigg|_0^\infty\right]
\end{aligned}$$

Assuming that $a - s < 0$ and $-a - s < 0$ (that is, $s > |a|$), the exponential functions tend to zero as $t \to \infty$.

$$\begin{aligned}
\mathcal{L}\{t^2 \sin at\} &= \frac{1}{2}\left[\frac{2}{(a-s)^3}(-1) - \frac{2}{(-a-s)^3}(-1)\right] \\
&= -\frac{1}{(a-s)^3} + \frac{1}{(-a-s)^3} \\
&= \frac{1}{(s-a)^3} - \frac{1}{(s+a)^3} \\
&= \frac{(s+a)^3 - (s-a)^3}{(s-a)^3(s+a)^3} \\
&= \frac{6as^2 + 2a^3}{[(s-a)(s+a)]^3} \\
&= 2a\frac{3s^2 + a^2}{(s^2 - a^2)^3}
\end{aligned}$$