Problem 29

Suppose that f and f' are continuous for $t \geq 0$ and of exponential order as $t \to \infty$. Use integration by parts to show that if $F(s) = \mathcal{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$. The result is actually true under less restrictive conditions, such as those of Theorem 6.1.2.

Solution

Using Integration by Parts

Use integration by parts in the definition of the Laplace transform.

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \\ &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^\infty \frac{d}{dt}\left(-\frac{1}{s}e^{-st}\right) f(t)\, dt \\ &= \left(-\frac{1}{s}e^{-st}\right) f(t)\bigg|_0^\infty - \int_0^\infty \left(-\frac{1}{s}e^{-st}\right) f'(t)\, dt \\ &= \frac{1}{s} f(0) + \frac{1}{s}\int_0^\infty e^{-st} f'(t)\, dt \end{aligned}$$

Consider the magnitude of $F(s)$.

$$|F(s)| = \left|\frac{1}{s} f(0) + \frac{1}{s}\int_0^\infty e^{-st} f'(t)\, dt\right|$$

Use the triangle inequality.

$$|F(s)| \leq \left|\frac{1}{s} f(0)\right| + \left|\frac{1}{s}\int_0^\infty e^{-st} f'(t)\, dt\right|$$

Take the limit of both sides as $s \to \infty$.

$$\lim_{s\to\infty} |F(s)| \leq \lim_{s\to\infty} \frac{|f(0)|}{s} + \lim_{s\to\infty} \left|\frac{1}{s}\int_0^\infty e^{-st} f'(t)\, dt\right|$$

As $f(0)$ is only a constant, the first limit is zero.

$$\begin{aligned} \lim_{s\to\infty} |F(s)| &\leq \lim_{s\to\infty} \left|\frac{1}{s}\int_0^\infty e^{-st} f'(t)\, dt\right| \\ &= \lim_{s\to\infty} \frac{1}{s}\left|\int_0^\infty e^{-st} f'(t)\, dt\right| \\ &\leq \lim_{s\to\infty} \frac{1}{s}\int_0^\infty \left|e^{-st} f'(t)\right| dt \end{aligned}$$

Since f' is of exponential order, $|f'(t)| \leq Ke^{at}$, where K is a real positive constant and a is a real constant.

$$\begin{aligned}
\lim_{s\to\infty} |F(s)| &\leq \lim_{s\to\infty} \frac{1}{s} \int_0^\infty \left| e^{-st} f'(t) \right| \, dt \\
&= \lim_{s\to\infty} \frac{1}{s} \int_0^\infty e^{-st} \left| f'(t) \right| \, dt \\
&\leq \lim_{s\to\infty} \frac{1}{s} \int_0^\infty e^{-st} K e^{at} \, dt \\
&= \lim_{s\to\infty} \frac{K}{s} \int_0^\infty e^{(a-s)t} \, dt \\
&= \lim_{s\to\infty} \frac{K}{s} \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty \\
&= \lim_{s\to\infty} \frac{K}{s} \frac{1}{s-a}
\end{aligned}$$

Evaluating the limit on the right side,

$$\lim_{s\to\infty} |F(s)| \leq 0.$$

Because the magnitude of a number cannot be negative, this limit must be zero.

$$\lim_{s\to\infty} |F(s)| = 0$$

The only number with a magnitude of zero is zero. Therefore,

$$\lim_{s\to\infty} F(s) = 0.$$

Without Using Integration by Parts

Start with the definition of the Laplace transform.

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \\ &= \int_0^\infty e^{-st} f(t)\, dt \end{aligned}$$

Consider the magnitude of $F(s)$.

$$|F(s)| = \left| \int_0^\infty e^{-st} f(t)\, dt \right|$$

Take the limit of both sides as $s \to \infty$.

$$\begin{aligned} \lim_{s\to\infty} |F(s)| &= \lim_{s\to\infty} \left| \int_0^\infty e^{-st} f(t)\, dt \right| \\ &\le \lim_{s\to\infty} \int_0^\infty \left| e^{-st} f(t) \right| dt \\ &= \lim_{s\to\infty} \int_0^\infty e^{-st} \, |f(t)|\, dt \end{aligned}$$

Since f is of exponential order, $|f(t)| \le Ke^{at}$, where K is a real positive constant and a is a real constant.

$$\begin{aligned} \lim_{s\to\infty} |F(s)| &\le \lim_{s\to\infty} \int_0^\infty e^{-st} K e^{at}\, dt \\ &= \lim_{s\to\infty} K \int_0^\infty e^{(a-s)t}\, dt \\ &= \lim_{s\to\infty} \frac{K}{a-s} e^{(a-s)t} \Big|_0^\infty \\ &= \lim_{s\to\infty} \frac{K}{s-a} \end{aligned}$$

Evaluating the limit on the right side,

$$\lim_{s\to\infty} |F(s)| \le 0.$$

Because the magnitude of a number cannot be negative, this limit must be zero.

$$\lim_{s\to\infty} |F(s)| = 0$$

The only number with a magnitude of zero is zero. Therefore,

$$\lim_{s\to\infty} F(s) = 0.$$