Problem 31

Consider the Laplace transform of t^p, where $p > -1$.

(a) Referring to Problem 30, show that

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st}t^p\,dt = \frac{1}{s^{p+1}}\int_0^\infty e^{-x}x^p\,dx$$
$$= \Gamma(p+1)/s^{p+1}, \qquad s > 0.$$

(b) Let p be a positive integer n in part (a); show that

$$\mathcal{L}\{t^n\} = n!/s^{n+1}, \qquad s > 0.$$

(c) Show that

$$\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}\,dx, \qquad s > 0.$$

It is possible to show that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2};$$

hence

$$\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi/s}, \qquad s > 0.$$

(d) Show that

$$\mathcal{L}\{t^{1/2}\} = \sqrt{\pi}/(2s^{3/2}), \qquad s > 0.$$

Solution

Part (a)

Start by using the definition of the Laplace transform.

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st}t^p\,dt$$

For this integral to converge, it must be the case that $s > 0$. Make the substitution $x = st$. Then $dx = s\,dt$.

$$\begin{aligned}\mathcal{L}\{t^p\} &= \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^p\frac{dx}{s}\\ &= \int_0^\infty e^{-x}\frac{x^p}{s^p}\frac{dx}{s}\\ &= \frac{1}{s^{p+1}}\int_0^\infty e^{-x}x^p\,dx\\ &= \frac{1}{s^{p+1}}\Gamma(p+1)\end{aligned}$$

Part (b)

Suppose now that p is a positive integer n.

$$\begin{aligned}
\mathcal{L}\{t^n\} &= \frac{1}{s^{n+1}}\Gamma(n+1) \\
&= \frac{1}{s^{n+1}}n\Gamma(n) \\
&= \frac{1}{s^{n+1}}n(n-1)\Gamma(n-1) \\
&= \frac{1}{s^{n+1}}n(n-1)(n-2)\Gamma(n-2) \\
&= \frac{1}{s^{n+1}}n(n-1)(n-2)\cdots(3)(2)\Gamma(2) \\
&= \frac{1}{s^{n+1}}n(n-1)(n-2)\cdots(3)(2)(1)\Gamma(1) \\
&= \frac{1}{s^{n+1}}n(n-1)(n-2)\cdots(3)(2)(1)(1) \\
&= \frac{1}{s^{n+1}}n(n-1)(n-2)\cdots(3)(2)(1) \\
&= \frac{1}{s^{n+1}}n!
\end{aligned}$$

Part (c)

Plug in $p = -1/2$ in the result of part (a).

$$\begin{aligned}
\mathcal{L}\{t^{-1/2}\} &= \frac{1}{s^{-\frac{1}{2}+1}}\Gamma\left(-\frac{1}{2}+1\right) \\
&= \frac{1}{s^{1/2}}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{1}{s^{1/2}}\sqrt{\pi} \\
&= \sqrt{\frac{\pi}{s}}
\end{aligned}$$

Alternatively, use the definition of the Laplace transform once again.

$$\mathcal{L}\{t^{-1/2}\} = \int_0^\infty e^{-st}t^{-1/2}\,dt$$

Make the substitution $x^2 = st$. Then $2x\,dx = s\,dt$.

$$\begin{aligned}
\mathcal{L}\{t^{-1/2}\} &= \int_0^\infty e^{-x^2}\left(\frac{x^2}{s}\right)^{-1/2}\frac{2x\,dx}{s} \\
&= \int_0^\infty e^{-x^2}\left(\frac{s}{x^2}\right)^{1/2}\frac{2x\,dx}{s} \\
&= \int_0^\infty e^{-x^2}\left(\frac{s^{1/2}}{x}\right)\frac{2x\,dx}{s} \\
&= \int_0^\infty e^{-x^2}\frac{2\,dx}{s^{1/2}} \\
&= \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}\,dx \\
&= \frac{2}{\sqrt{s}}\frac{\sqrt{\pi}}{2} \\
&= \sqrt{\frac{\pi}{s}}
\end{aligned}$$

Part (d)

Plug in $p = 1/2$ in the result of part (a).

$$\begin{aligned}
\mathcal{L}\{t^{1/2}\} &= \frac{1}{s^{\frac{1}{2}+1}}\Gamma\left(\frac{1}{2}+1\right) \\
&= \frac{1}{s^{3/2}}\Gamma\left(\frac{3}{2}\right) \\
&= \frac{1}{s^{3/2}}\frac{\sqrt{\pi}}{2} \\
&= \frac{\sqrt{\pi}}{2s^{3/2}}
\end{aligned}$$

Alternatively, we can use the following property of the Laplace transform.

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s)$$

Doing so gives

$$\begin{aligned}
\mathcal{L}\{t^{1/2}\} &= \mathcal{L}\{t\cdot t^{-1/2}\} \\
&= -\frac{d}{ds}\mathcal{L}\{t^{-1/2}\} \\
&= -\frac{d}{ds}\left(\sqrt{\frac{\pi}{s}}\right) \\
&= -\left(-\frac{\sqrt{\pi}}{2s^{3/2}}\right) \\
&= \frac{\sqrt{\pi}}{2s^{3/2}}.
\end{aligned}$$