Problem 31

Consider the Laplace transform of t^p, where $p > -1$.

(a) Referring to Problem 30, show that

$$\begin{aligned}\mathcal{L}\{t^p\} &= \int_0^\infty e^{-st}t^p\,dt = \frac{1}{s^{p+1}}\int_0^\infty e^{-x}x^p\,dx \\ &= \Gamma(p+1)/s^{p+1}, \qquad s > 0.\end{aligned}$$

(b) Let p be a positive integer n in part (a); show that

$$\mathcal{L}\{t^n\} = n!/s^{n+1}, \qquad s > 0.$$

(c) Show that

$$\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}}\int_0^\infty e^{-x^2}\,dx, \qquad s > 0.$$

It is possible to show that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2};$$

hence

$$\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi/s}, \qquad s > 0.$$

(d) Show that

$$\mathcal{L}\{t^{1/2}\} = \sqrt{\pi}/(2s^{3/2}), \qquad s > 0.$$