Problem 8

Recall that $\cosh bt = (e^{bt} + e^{-bt})/2$ and $\sinh bt = (e^{bt} - e^{-bt})/2$. In each of Problems 7 through 10, find the Laplace transform of the given function; a and b are real constants.

$$f(t) = \sinh bt$$

Solution

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Find the Laplace transform of $\sinh bt$.

$$\begin{aligned}
\mathcal{L}\{\sinh bt\} &= \int_0^\infty e^{-st} \sinh bt\, dt \\
&= \int_0^\infty e^{-st} \frac{e^{bt} - e^{-bt}}{2}\, dt \\
&= \frac{1}{2} \int_0^\infty e^{-st}(e^{bt} - e^{-bt})\, dt \\
&= \frac{1}{2} \left(\int_0^\infty e^{bt-st}\, dt - \int_0^\infty e^{-bt-st}\, dt \right) \\
&= \frac{1}{2} \left[\int_0^\infty e^{(b-s)t}\, dt - \int_0^\infty e^{(-b-s)t}\, dt \right] \\
&= \frac{1}{2} \left[\frac{1}{b-s} e^{(b-s)t} \Big|_0^\infty - \frac{1}{-b-s} e^{(-b-s)t} \Big|_0^\infty \right] \\
&= \frac{1}{2} \left[\frac{1}{b-s}(-1) - \frac{1}{-b-s}(-1) \right] \qquad (1) \\
&= \frac{1}{2} \left(\frac{1}{s-b} - \frac{1}{s+b} \right) \\
&= \frac{1}{2} \left[\frac{s+b-(s-b)}{(s-b)(s+b)} \right] \\
&= \frac{1}{2} \left(\frac{2b}{s^2 - b^2} \right) \\
&= \frac{b}{s^2 - b^2}
\end{aligned}$$

Note that for equation (1) to hold, it is critical that $b - s < 0$ and $-b - s < 0$, that is,

$$s > b \quad \text{and} \quad s > -b$$

$$s > |b|.$$