

## Problem 14

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

$$y'' - 4y' + 4y = 0; \quad y(0) = 1, \quad y'(0) = 1$$

### Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function  $y(t)$  is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^{\infty} e^{-st}y(t) dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' - 4y' + 4y\} = \mathcal{L}\{0\}$$

Use the fact that the transform is a linear operator.

$$\begin{aligned} \mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} &= 0 \\ [s^2Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + 4Y(s) &= 0 \end{aligned}$$

Plug in the initial conditions,  $y(0) = 1$  and  $y'(0) = 1$ .

$$[s^2Y(s) - s - 1] - 4[sY(s) - 1] + 4Y(s) = 0$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for  $Y$ , the transformed solution.

$$\begin{aligned} s^2Y(s) - 4sY(s) + 4Y(s) - s - 1 + 4 &= 0 \\ (s^2 - 4s + 4)Y(s) &= s - 3 \\ Y(s) &= \frac{s - 3}{s^2 - 4s + 4} \\ &= \frac{s - 3}{(s - 2)^2} \\ &= \frac{1}{s - 2} + \frac{-1}{(s - 2)^2} \end{aligned}$$

Take the inverse Laplace transform of  $Y(s)$  now to recover  $y(t)$ .

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s - 2} + \frac{-1}{(s - 2)^2}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s - 2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s - 2)^2}\right\} \\ &= e^{2t} - te^{2t} \end{aligned}$$

