Problem 15

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

$$y'' - 2y' + 4y = 0; \qquad y(0) = 2, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' - 2y' + 4y\} = \mathcal{L}\{0\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 0$$
$$[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 4Y(s) = 0$$

Plug in the initial conditions, $y(0) = 2$ and $y'(0) = 0$.

$$[s^2Y(s) - 2s] - 2[sY(s) - 2] + 4Y(s) = 0$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$s^2Y(s) - 2sY(s) + 4Y(s) - 2s + 4 = 0$$
$$(s^2 - 2s + 4)Y(s) = 2s - 4$$

$$\begin{aligned}
Y(s) &= \frac{2s-4}{s^2-2s+4} \\
&= \frac{2s-4}{s^2-2s+1+4-1} \\
&= \frac{2s-4}{(s-1)^2+3} \\
&= \frac{2s-2-4+2}{(s-1)^2+3} \\
&= \frac{2(s-1)-2}{(s-1)^2+3} \\
&= 2\frac{s-1}{(s-1)^2+3} - \frac{2}{(s-1)^2+3} \\
&= 2\frac{s-1}{(s-1)^2+3} - \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{(s-1)^2+3}
\end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{2\frac{s-1}{(s-1)^2+3} - \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{(s-1)^2+3}\right\} \\ &= 2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+3}\right\} - \frac{2}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s-1)^2+3}\right\} \\ &= 2e^t\cos\sqrt{3}t - \frac{2}{\sqrt{3}}e^t\sin\sqrt{3}t \end{aligned}$$