

## Problem 16

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

$$y'' + 2y' + 5y = 0; \quad y(0) = 2, \quad y'(0) = -1$$

### Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function  $y(t)$  is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^{\infty} e^{-st} y(t) dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 2y' + 5y\} = \mathcal{L}\{0\}$$

Use the fact that the transform is a linear operator.

$$\begin{aligned} \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} &= 0 \\ [s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 5Y(s) &= 0 \end{aligned}$$

Plug in the initial conditions,  $y(0) = 2$  and  $y'(0) = -1$ .

$$[s^2Y(s) - 2s + 1] + 2[sY(s) - 2] + 5Y(s) = 0$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for  $Y$ , the transformed solution.

$$\begin{aligned} s^2Y(s) + 2sY(s) + 5Y(s) - 2s + 1 - 4 &= 0 \\ (s^2 + 2s + 5)Y(s) &= 2s + 3 \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{2s + 3}{s^2 + 2s + 5} \\ &= \frac{2s + 3}{s^2 + 2s + 1 + 5 - 1} \\ &= \frac{2s + 3}{(s + 1)^2 + 4} \\ &= \frac{2s + 2 + 3 - 2}{(s + 1)^2 + 4} \\ &= \frac{2(s + 1) + 1}{(s + 1)^2 + 4} \\ &= 2 \frac{s + 1}{(s + 1)^2 + 4} + \frac{1}{(s + 1)^2 + 4} \\ &= 2 \frac{s + 1}{(s + 1)^2 + 4} + \frac{1}{2} \frac{2}{(s + 1)^2 + 4} \end{aligned}$$

Take the inverse Laplace transform of  $Y(s)$  now to recover  $y(t)$ .

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\&= \mathcal{L}^{-1}\left\{2\frac{s+1}{(s+1)^2+4} + \frac{1}{2}\frac{2}{(s+1)^2+4}\right\} \\&= 2\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} \\&= 2e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t\end{aligned}$$

