Problem 17

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

$$y^{(4)} - 4y''' + 6y'' - 4y' + y = 0; \qquad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 1$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the derivatives transform as follows.

$$\begin{aligned}
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)
\end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y^{(4)} - 4y''' + 6y'' - 4y' + y\} = \mathcal{L}\{0\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y^{(4)}\} - 4\mathcal{L}\{y'''\} + 6\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 0$$

$$\begin{aligned}
[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] - 4[s^3Y(s) - s^2y(0) - sy'(0) - y''(0)] \\
+ 6[s^2Y(s) - sy(0) - y'(0)] - 4[sY(s) - y(0)] + Y(s) = 0
\end{aligned}$$

Plug in the initial conditions, $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, and $y'''(0) = 1$.

$$[s^4Y(s) - s^2 - 1] - 4[s^3Y(s) - s] + 6[s^2Y(s) - 1] - 4[sY(s)] + Y(s) = 0$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$s^4Y(s) - 4s^3Y(s) + 6s^2Y(s) - 4sY(s) + Y(s) - s^2 - 1 + 4s - 6 = 0$$

$$(s^4 - 4s^3 + 6s^2 - 4s + 1)Y(s) = s^2 - 4s + 7$$

$$\begin{aligned}
Y(s) &= \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2 - 4s + 1} \\
&= \frac{s^2 - 4s + 7}{(s-1)^4} \\
&= \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4} \\
&= \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{6}\frac{6}{(s-1)^4}
\end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{6}\frac{6}{(s-1)^4}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\} + \frac{4}{6}\mathcal{L}^{-1}\left\{\frac{6}{(s-1)^4}\right\} \\
&= te^t - t^2e^t + \frac{4}{6}t^3e^t \\
&= \frac{1}{3}te^t\left(3 - 3t + 2t^2\right)
\end{aligned}$$