

Problem 18

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

$$y^{(4)} - y = 0; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^{\infty} e^{-st}y(t) dt.$$

Consequently, the derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\ \mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\ \mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) \end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y^{(4)} - y\} = \mathcal{L}\{0\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = 0$$

$$[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] - Y(s) = 0$$

Plug in the initial conditions, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$, and $y'''(0) = 0$.

$$[s^4Y(s) - s^3 - s] - Y(s) = 0$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y , the transformed solution.

$$s^4Y(s) - Y(s) - s^3 - s = 0$$

$$(s^4 - 1)Y(s) = s^3 + s$$

$$\begin{aligned} Y(s) &= \frac{s^3 + s}{s^4 - 1} \\ &= \frac{s(s^2 + 1)}{(s + 1)(s - 1)(s^2 + 1)} \\ &= \frac{s}{(s + 1)(s - 1)} \\ &= \frac{1/2}{s + 1} + \frac{1/2}{s - 1} \end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\&= \mathcal{L}^{-1}\left\{\frac{1/2}{s+1} + \frac{1/2}{s-1}\right\} \\&= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \\&= \frac{1}{2}e^{-t} + \frac{1}{2}e^t \\&= \frac{e^{-t} + e^t}{2} \\&= \cosh t\end{aligned}$$

