Problem 21

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

$$y'' - 2y' + 2y = \cos t; \qquad y(0) = 1, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned}
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0)
\end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{\cos t\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{\cos t\}$$

$$[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 2Y(s) = \frac{s}{s^2+1}$$

Plug in the initial conditions, $y(0) = 1$ and $y'(0) = 0$.

$$[s^2Y(s) - s] - 2[sY(s) - 1] + 2Y(s) = \frac{s}{s^2+1}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$s^2Y(s) - 2sY(s) + 2Y(s) - s + 2 = \frac{s}{s^2+1}$$

$$(s^2 - 2s + 2)Y(s) = \frac{s}{s^2+1} + s - 2$$

Divide both sides by $s^2 - 2s + 2$.

$$\begin{aligned}
Y(s) &= \frac{s}{(s^2+1)(s^2-2s+2)} + \frac{s-2}{s^2-2s+2} \\
&= \frac{\frac{1}{5}s - \frac{2}{5}}{s^2+1} + \frac{-\frac{1}{5}s + \frac{4}{5}}{s^2-2s+2} + \frac{s-2}{s^2-2s+2} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{s-4}{s^2-2s+2} + \frac{s-2}{s^2-2s+2}
\end{aligned}$$

Complete the square in the denominator of the last two terms.

$$\begin{aligned}
Y(s) &= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{s-4}{s^2-2s+1+2-1} + \frac{s-2}{s^2-2s+1+2-1} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{s-4}{(s-1)^2+1} + \frac{s-2}{(s-1)^2+1} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{s-1-4+1}{(s-1)^2+1} + \frac{s-1-2+1}{(s-1)^2+1} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{s-1-3}{(s-1)^2+1} + \frac{s-1-1}{(s-1)^2+1} \\
&= \frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1} + \frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1}
\end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{s}{s^2+1} - \frac{2}{5}\frac{1}{s^2+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1} + \frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1}\right\} \\
&= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} + \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} \\
&\qquad + \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} \\
&= \frac{1}{5}\cos t - \frac{2}{5}\sin t - \frac{1}{5}e^t\cos t + \frac{3}{5}e^t\sin t + e^t\cos t - e^t\sin t \\
&= \frac{1}{5}\cos t - \frac{2}{5}\sin t + \frac{4}{5}e^t\cos t - \frac{2}{5}e^t\sin t
\end{aligned}$$