Problem 22

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

$$y'' - 2y' + 2y = e^{-t}; \qquad y(0) = 0, \quad y'(0) = 1$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{e^{-t}\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+1}$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 1$.

$$[s^2Y(s) - 1] - 2[sY(s)] + 2Y(s) = \frac{1}{s+1}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$s^2Y(s) - 2sY(s) + 2Y(s) - 1 = \frac{1}{s+1}$$

$$(s^2 - 2s + 2)Y(s) = \frac{1}{s+1} + 1$$
$$= \frac{s+2}{s+1}$$

Divide both sides by $s^2 - 2s + 2$.

$$Y(s) = \frac{s+2}{(s+1)(s^2 - 2s + 2)}$$

Use partial fraction decomposition to write $Y(s)$ in terms of known transforms.

$$\frac{s+2}{(s+1)(s^2-2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2}$$

Multiply both sides by $(s+1)(s^2-2s+2)$.

$$s+2 = A(s^2-2s+2) + (Bs+C)(s+1)$$

Choose three random values for s to get a system of equations for A, B, and C.

$$\begin{aligned} s=0: &\qquad 2 = 2A + C \\ s=1: &\qquad 3 = A + 2B + 2C \\ s=2: &\qquad 4 = 2A + 6B + 3C \end{aligned}$$

Solving this system yields

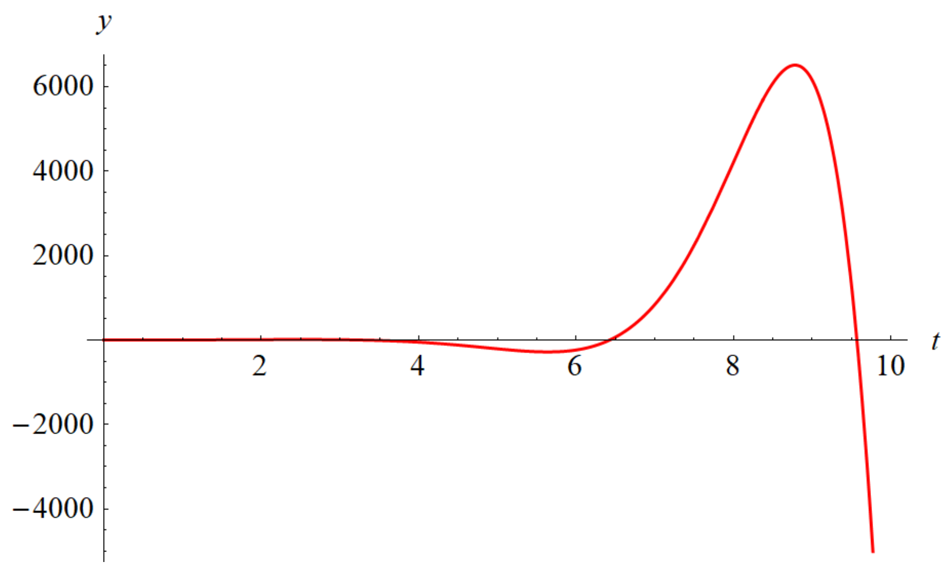
$$A = \frac{1}{5} \quad \text{and} \quad B = -\frac{1}{5} \quad \text{and} \quad C = \frac{8}{5},$$

so $Y(s)$ becomes

$$\begin{aligned} Y(s) &= \frac{\frac{1}{5}}{s+1} + \frac{-\frac{1}{5}s + \frac{8}{5}}{s^2-2s+2} \\ &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-8}{s^2-2s+2} \\ &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-8}{s^2-2s+1+2-1} \\ &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-8}{(s-1)^2+1} \\ &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-1-8+1}{(s-1)^2+1} \\ &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-1-7}{(s-1)^2+1} \\ &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{7}{5}\frac{1}{(s-1)^2+1}. \end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{7}{5}\frac{1}{(s-1)^2+1}\right\} \\ &= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\} \\ &= \frac{1}{5}e^{-t} - \frac{1}{5}e^t\cos t + \frac{7}{5}e^t\sin t \\ &= \frac{1}{5}[e^{-t} + e^t(7\sin t - \cos t)] \end{aligned}$$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