Problem 23

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

$$y'' + 2y' + y = 4e^{-t}; \qquad y(0) = 2, \quad y'(0) = -1$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 2y' + y\} = \mathcal{L}\{4e^{-t}\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{4e^{-t}\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + Y(s) = \frac{4}{s+1}$$

Plug in the initial conditions, $y(0) = 2$ and $y'(0) = -1$.

$$[s^2Y(s) - 2s + 1] + 2[sY(s) - 2] + Y(s) = \frac{4}{s+1}$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$s^2Y(s) + 2sY(s) + Y(s) - 2s + 1 - 4 = \frac{4}{s+1}$$

$$(s^2 + 2s + 1)Y(s) = \frac{4}{s+1} + 2s + 3$$

Divide both sides by $s^2 + 2s + 1$.

$$\begin{aligned} Y(s) &= \frac{4}{(s+1)(s^2+2s+1)} + \frac{2s+3}{s^2+2s+1} \\ &= \frac{4}{(s+1)^3} + \frac{2s+3}{(s+1)^2} \\ &= 2\frac{2}{(s+1)^3} + \frac{2}{s+1} + \frac{1}{(s+1)^2} \end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{2\frac{2}{(s+1)^3} + \frac{2}{s+1} + \frac{1}{(s+1)^2}\right\} \\
&= 2\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^3}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \\
&= 2t^2e^{-t} + 2e^{-t} + te^{-t} \\
&= e^{-t}(2t^2 + t + 2)
\end{aligned}$$