Problem 29

Let

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt.$$

It is possible to show that as long as f satisfies the conditions of Theorem 6.1.2, it is legitimate to differentiate under the integral sign with respect to the parameter s when $s > a$.

(a) Show that $F'(s) = \mathcal{L}\{-tf(t)\}$.

(b) Show that $F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}$; hence differentiating the Laplace transform corresponds to multiplying the original function by $-t$.

Solution

Part (a)

Differentiate both sides with respect to s.

$$\begin{aligned} F'(s) &= \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt \\ &= \int_0^\infty (-te^{-st}) f(t)\,dt \\ &= \int_0^\infty e^{-st}[-tf(t)]\,dt \\ &= \mathcal{L}\{-tf(t)\} \end{aligned}$$

Part (b)

Differentiate both sides n times with respect to s.

$$\begin{aligned} F^{(n)}(s) &= \frac{d^n}{ds^n}\int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^\infty \frac{\partial^n}{\partial s^n} e^{-st} f(t)\,dt \\ &= \int_0^\infty [(-t)^n e^{-st}] f(t)\,dt \\ &= \int_0^\infty e^{-st}[(-t)^n f(t)]\,dt \\ &= \mathcal{L}\{(-t)^n f(t)\} \end{aligned}$$