Problem 31

In each of Problems 30 through 35, use the result of Problem 29 to find the Laplace transform of the given function; a and b are real numbers and n is a positive integer.

$$f(t) = t^2 \sin bt$$

Solution

The Laplace transform of a function $f(t)$ is defined here as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Substitute the given function and evaluate the integral.

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} t^2 \sin bt\,dt \\
&= \int_0^\infty \left(\frac{\partial^2}{\partial s^2} e^{-st}\right) \sin bt\,dt \\
&= \frac{d^2}{ds^2} \int_0^\infty e^{-st} \sin bt\,dt \\
&= \frac{d^2}{ds^2} \mathcal{L}\{\sin bt\} \\
&= \frac{d^2}{ds^2} \left(\frac{b}{s^2 + b^2}\right) \\
&= \frac{d}{ds} \left[-\frac{2bs}{(s^2 + b^2)^2}\right] \\
&= -\frac{(2b)(s^2 + b^2)^2 - 2bs[2(s^2 + b^2)2s]}{(s^2 + b^2)^4} \\
&= -\frac{2b(s^4 + 2s^2b^2 + b^4) - 8bs^2(s^2 + b^2)}{(s^2 + b^2)^4} \\
&= -\frac{2b^5 - 4b^3s^2 - 6bs^4}{(s^2 + b^2)^4} \\
&= -\frac{2b(b^2 - 3s^2)(s^2 + b^2)}{(s^2 + b^2)^4} \\
&= \frac{2b(3s^2 - b^2)}{(s^2 + b^2)^3}
\end{aligned}$$