Problem 34

In each of Problems 30 through 35, use the result of Problem 29 to find the Laplace transform of the given function; a and b are real numbers and n is a positive integer.

$$f(t) = te^{at}\sin bt$$

Solution

The Laplace transform of a function $f(t)$ is defined here as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt.$$

Substitute the given function and evaluate the integral.

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st}te^{at}\sin bt\,dt \\
&= \int_0^\infty \left(-\frac{\partial}{\partial s}e^{-st}\right)e^{at}\sin bt\,dt \\
&= -\frac{d}{ds}\int_0^\infty e^{-st}e^{at}\sin bt\,dt \\
&= -\frac{d}{ds}\mathcal{L}\{e^{at}\sin bt\} \\
&= -\frac{d}{ds}\left[\frac{b}{(s-a)^2+b^2}\right] \\
&= -\frac{-b[2(s-a)]}{[(s-a)^2+b^2]^2} \\
&= \frac{2b(s-a)}{[(s-a)^2+b^2]^2}
\end{aligned}$$