

Problem 35

In each of Problems 30 through 35, use the result of Problem 29 to find the Laplace transform of the given function; a and b are real numbers and n is a positive integer.

$$f(t) = te^{at} \cos bt$$

Solution

The Laplace transform of a function $f(t)$ is defined here as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

Substitute the given function and evaluate the integral.

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} t e^{at} \cos bt dt \\ &= \int_0^{\infty} \left(-\frac{\partial}{\partial s} e^{-st} \right) e^{at} \cos bt dt \\ &= -\frac{d}{ds} \int_0^{\infty} e^{-st} e^{at} \cos bt dt \\ &= -\frac{d}{ds} \mathcal{L}\{e^{at} \cos bt\} \\ &= -\frac{d}{ds} \left[\frac{s-a}{(s-a)^2 + b^2} \right] \\ &= -\frac{1[(s-a)^2 + b^2] - (s-a)[2(s-a)]}{[(s-a)^2 + b^2]^2} \\ &= -\frac{b^2 - (s-a)^2}{[(s-a)^2 + b^2]^2} \\ &= \frac{(s-a)^2 - b^2}{[(s-a)^2 + b^2]^2} \end{aligned}$$