Problem 36

Consider Bessel's equation of order zero

$$ty'' + y' + ty = 0.$$

Recall from Section 5.7 that $t = 0$ is a regular singular point for this equation, and therefore solutions may become unbounded as $t \to 0$. However, let us try to determine whether there are any solutions that remain finite at $t = 0$ and have finite derivatives there. Assuming that there is such a solution $y = \phi(t)$, let $Y(s) = \mathcal{L}\{\phi(t)\}$.

(a) Show that $Y(s)$ satisfies

$$(1 + s^2)Y'(s) + sY(s) = 0.$$

(b) Show that $Y(s) = c(1 + s^2)^{-1/2}$, where c is an arbitrary constant.

(c) Writing $(1 + s^2)^{-1/2} = s^{-1}(1 + s^{-2})^{-1/2}$, expanding in a binomial series valid for $s > 1$, and assuming that it is permissible to take the inverse transform term by term, show that

$$y = c\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = cJ_0(t),$$

where J_0 is the Bessel function of the first kind of order zero. Note that $J_0(0) = 1$ and that J_0 has finite derivatives of all orders at $t = 0$. It was shown in Section 5.7 that the second solution of this equation becomes unbounded as $t \to 0$.

Solution

Part (a)

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{ty'' + y' + ty\} = \mathcal{L}\{0\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{ty''\} + \mathcal{L}\{y'\} + \mathcal{L}\{ty\} = \mathcal{L}\{0\}$$

Apply the property, $(-d/ds)F(s) = \mathcal{L}\{tf(t)\}$.

$$-\frac{d}{ds}\mathcal{L}\{y''\} + \mathcal{L}\{y'\} - \frac{d}{ds}\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$-\frac{d}{ds}[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] - \frac{d}{ds}Y(s) = 0$$

$$-[2sY(s) + s^2Y'(s) - y(0)] + [sY(s) - y(0)] - Y'(s) = 0$$

$$-2sY(s) - s^2Y'(s) + \cancel{y(0)} + sY(s) - \cancel{y(0)} - Y'(s) = 0$$

$$(s^2+1)Y'(s) + sY(s) = 0$$

Part (b)

Solve for $Y'(s)/Y(s)$.

$$\frac{Y'(s)}{Y(s)} = -\frac{s}{s^2+1}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{ds}\ln|Y(s)| = -\frac{s}{s^2+1}$$

Integrate both sides with respect to s.

$$\begin{aligned}\ln|Y(s)| &= -\frac{1}{2}\ln(s^2+1) + C_1 \\ &= \ln(s^2+1)^{-1/2} + C_1\end{aligned}$$

Exponentiate both sides.

$$\begin{aligned}|Y(s)| &= e^{\ln(s^2+1)^{-1/2} + C_1} \\ &= e^{\ln(s^2+1)^{-1/2}}e^{C_1} \\ &= e^{C_1}(s^2+1)^{-1/2}\end{aligned}$$

Place $\pm$ on the right side to remove the absolute value sign on the left.

$$Y(s) = \pm e^{C_1}(s^2+1)^{-1/2}$$

Therefore, using a new constant c for $\pm e^{C_1}$,

$$Y(s) = c(s^2+1)^{-1/2}.$$

Part (c)

Now take the inverse Laplace transform of $Y(s)$ to recover $y(t)$.

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{c(s^2+1)^{-1/2}\right\} \\
&= \mathcal{L}^{-1}\left\{cs^{-1}(1+s^{-2})^{-1/2}\right\} \\
&= \mathcal{L}^{-1}\left\{cs^{-1}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(s^{-2})^n\right\} \\
&= \mathcal{L}^{-1}\left\{cs^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{2}\right)^n(2n-1)!!(s^{-2})^n\right\} \\
&= \mathcal{L}^{-1}\left\{cs^{-1}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(-1)^n}{2^n}\left[\frac{(2n)!}{2^n n!}\right](s^{-2})^n\right\} \\
&= \mathcal{L}^{-1}\left\{cs^{-1}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}(s^{-2})^n\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{c}{s}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}\cdot\frac{1}{s^{2n}}\right\} \\
&= c\mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}\cdot\frac{1}{s^{2n+1}}\right\} \\
&= c\sum_{n=0}^{\infty}\mathcal{L}^{-1}\left\{(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}\cdot\frac{1}{s^{2n+1}}\right\} \\
&= c\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}\mathcal{L}^{-1}\left\{\frac{(2n)!}{s^{2n+1}}\right\} \\
&= c\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n!)^2}t^{2n} \\
&= cJ_0(t)
\end{aligned}$$