Problem 37

For each of the following initial value problems, use the results of Problem 29 to find the differential equation satisfied by $Y(s) = \mathcal{L}\{\phi(t)\}$, where $y = \phi(t)$ is the solution of the given initial value problem.

(a) $y'' - ty = 0; \qquad y(0) = 1, \quad y'(0) = 0$ (Airy's equation)

(b) $(1-t^2)y'' - 2ty' + \alpha(\alpha+1)y = 0; \qquad y(0) = 0, \quad y'(0) = 1$ (Legendre's equation)

Note that the differential equation for $Y(s)$ is of first order in part (a), but of second order in part (b). This is due to the fact that t appears at most to the first power in the equation of part (a), whereas it appears to the second power in that of part (b). This illustrates that the Laplace transform is not often useful in solving differential equations with variable coefficients, unless all the coefficients are at most linear functions of the independent variable.

Solution

Because the ODEs are linear, the Laplace transform can be applied to solve them. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\,dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Part (a)

Take the Laplace transform of both sides of the Airy equation.

$$\mathcal{L}\{y'' - ty\} = \mathcal{L}\{0\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} - \mathcal{L}\{ty\} = \mathcal{L}\{0\}$$

For the second term on the left, use the property that $(-d/ds)Y(s) = \mathcal{L}\{ty(t)\}$.

$$[s^2Y(s) - sy(0) - y'(0)] + \frac{d}{ds}Y(s) = 0$$

Plug in the initial conditions, $y(0) = 1$ and $y'(0) = 0$.

$$[s^2Y(s) - s] + \frac{d}{ds}Y(s) = 0$$

$$Y'(s) + s^2Y(s) = s$$

To solve this first-order linear inhomogeneous equation, use the integrating factor,

$$I = \exp\left(\int^s r^2\,dr\right) = e^{s^3/3}.$$

Multiply both sides of the ODE by I.

$$e^{s^3/3}Y'(s) + s^2 e^{s^3/3}Y(s) = se^{s^3/3}$$

The left side can be written as $d/ds(IY)$ by the product rule.

$$\frac{d}{ds}\left[e^{s^3/3}Y(s)\right] = se^{s^3/3}$$

Integrate both sides with respect to s.

$$e^{s^3/3}Y(s) = \int^s re^{r^3/3}\,dr + C_1$$

Therefore, the transformed solution is

$$Y(s) = e^{-s^3/3}\left(\int^s re^{r^3/3}\,dr + C_1\right).$$

All that's left is to take the inverse Laplace transform of $Y(s)$ to recover $y(t)$.

Part (b)

Take the Laplace transform of both sides of the Legendre equation.

$$\mathcal{L}\{(1-t^2)y'' - 2ty' + \alpha(\alpha+1)y\} = \mathcal{L}\{0\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} - \mathcal{L}\{t^2y''\} - 2\mathcal{L}\{ty'\} + \alpha(\alpha+1)\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

For the second and third terms on the left, use the property that $(-1)^n(d^n/ds^n)Y(s) = \mathcal{L}\{t^n y(t)\}$.

$$\mathcal{L}\{y''\} - \frac{d^2}{ds^2}\mathcal{L}\{y''\} + 2\frac{d}{ds}\mathcal{L}\{y'\} + \alpha(\alpha+1)\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$[s^2Y(s) - sy(0) - y'(0)] - \frac{d^2}{ds^2}[s^2Y(s) - sy(0) - y'(0)] + 2\frac{d}{ds}[sY(s) - y(0)] + \alpha(\alpha+1)Y(s) = 0$$

Evaluate the derivatives.

$$[s^2Y(s) - sy(0) - y'(0)] - [2Y(s) + 4sY'(s) + s^2Y''(s)] + 2[Y(s) + sY'(s)] + \alpha(\alpha+1)Y(s) = 0$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 1$.

$$[s^2Y(s) - 1] - [2Y(s) + 4sY'(s) + s^2Y''(s)] + 2[Y(s) + sY'(s)] + \alpha(\alpha+1)Y(s) = 0$$

Therefore, the ODE that the transformed solution satisfies is

$$-s^2Y''(s) - 2sY'(s) + [\alpha(\alpha+1) + s^2]Y(s) = 1.$$