Problem 38

Suppose that

$$g(t) = \int_0^t f(\tau)\, d\tau.$$

If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that

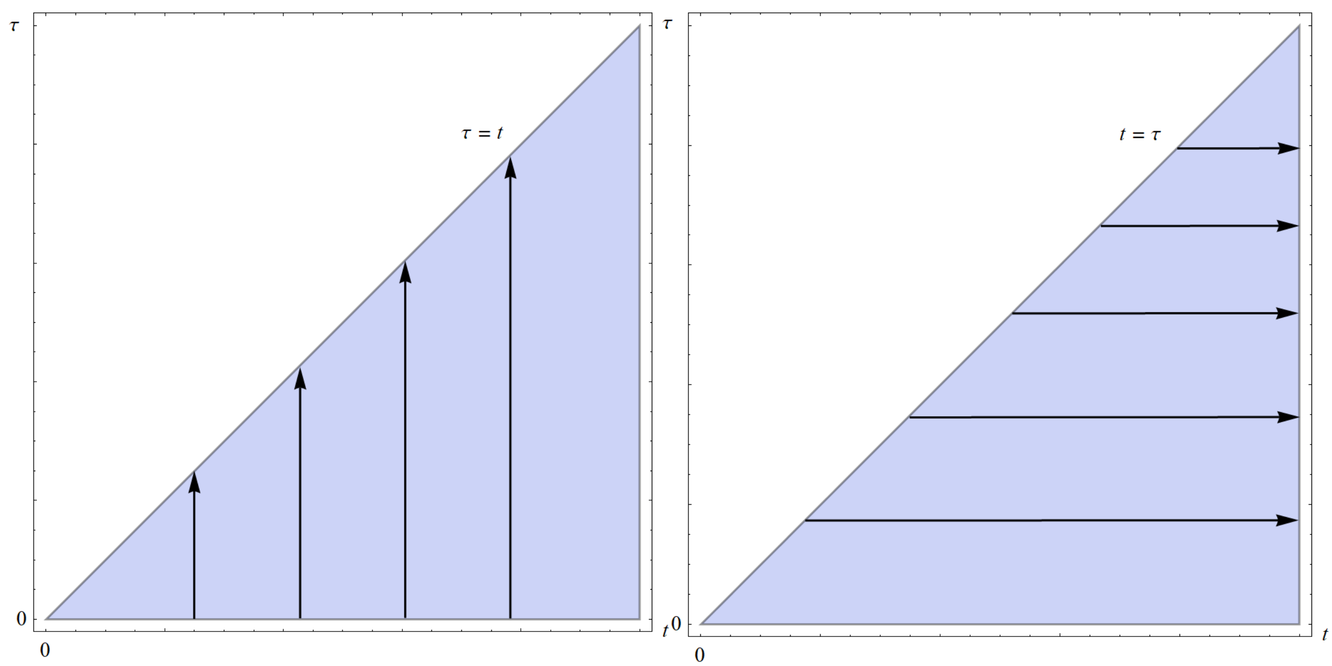
$$G(s) = F(s)/s.$$

Solution

Take the Laplace transform of both sides of the definition for $g(t)$.

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} \\
G(s) &= \int_0^\infty e^{-st}\left[\int_0^t f(\tau)\, d\tau\right] dt \\
&= \int_0^\infty \int_0^t e^{-st} f(\tau)\, d\tau\, dt
\end{aligned}$$

The current mode of integration in the $t\tau$-plane is shown below on the left.



Integrate over this domain as shown on the right to switch the order of integration.

$$\begin{aligned}
G(s) &= \int_0^\infty \int_\tau^\infty e^{-st} f(\tau)\, dt\, d\tau \\
&= \int_0^\infty \left(-\frac{1}{s}e^{-st}\right)\bigg|_\tau^\infty f(\tau)\, d\tau \\
&= \int_0^\infty \left(\frac{1}{s}e^{-s\tau}\right) f(\tau)\, d\tau \\
&= \frac{1}{s}\int_0^\infty e^{-s\tau} f(\tau)\, d\tau \\
&= \frac{1}{s}\mathcal{L}\{f(t)\} \\
&= \frac{F(s)}{s}
\end{aligned}$$