Problem 11

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

$$y'' - y' - 6y = 0; \qquad y(0) = 1, \quad y'(0) = -1$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\ \mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' - y' - 6y\} = \mathcal{L}\{0\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 0$$

$$[s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 6Y(s) = 0$$

Plug in the initial conditions, $y(0) = 1$ and $y'(0) = -1$.

$$[s^2Y(s) - s + 1] - [sY(s) - 1] - 6Y(s) = 0$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$s^2Y(s) - sY(s) - 6Y(s) - s + 2 = 0$$

$$(s^2 - s - 6)Y(s) = s - 2$$

$$\begin{aligned} Y(s) &= \frac{s-2}{s^2 - s - 6} \\ &= \frac{s-2}{(s-3)(s+2)} \\ &= \frac{1/5}{s-3} + \frac{4/5}{s+2} \end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1/5}{s-3} + \frac{4/5}{s+2}\right\} \\ &= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \\ &= \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t} \end{aligned}$$

Therefore,

$$y(t) = \frac{1}{5}(e^{3t} + 4e^{-2t}).$$