Problem 13

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

$$y'' - 2y' + 2y = 0; \qquad y(0) = 0, \quad y'(0) = 1$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' - 2y' + 2y\} = \mathcal{L}\{0\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 0$$

$$[s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 2Y(s) = 0$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 1$.

$$[s^2Y(s) - 1] - 2[sY(s)] + 2Y(s) = 0$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

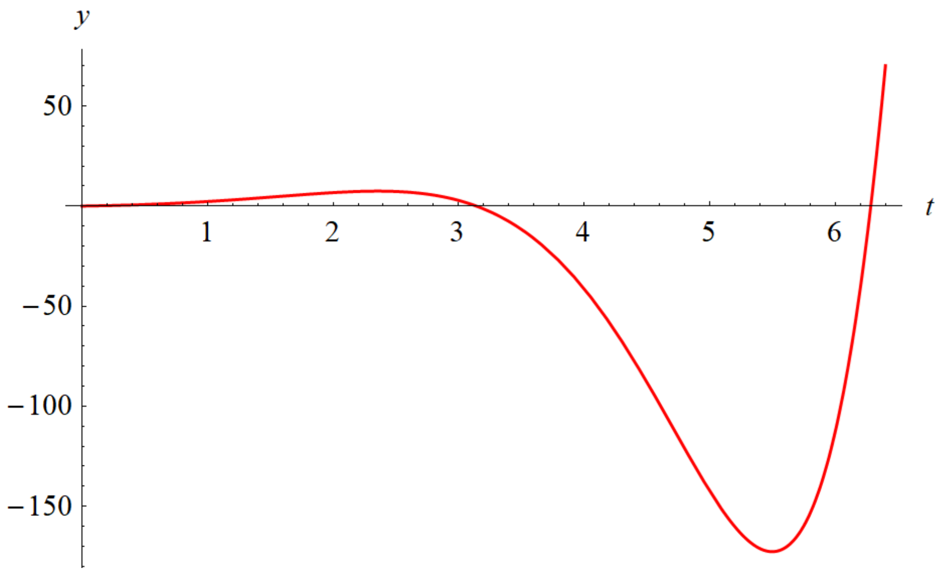
$$s^2Y(s) - 2sY(s) + 2Y(s) - 1 = 0$$

$$(s^2 - 2s + 2)Y(s) = 1$$

$$\begin{aligned} Y(s) &= \frac{1}{s^2 - 2s + 2} \\ &= \frac{1}{s^2 - 2s + 1 + 2 - 1} \\ &= \frac{1}{(s-1)^2 + 1} \end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2 + 1}\right\} \\ &= e^t \sin t \end{aligned}$$


y
50
1
2
3
4
5
6
t
−50
−100
−150