Problem 19

In each of Problems 11 through 23, use the Laplace transform to solve the given initial value problem.

$$y^{(4)} - 4y = 0; \qquad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\,dt.$$

Consequently, the derivatives transform as follows.

$$\begin{aligned}
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathcal{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
\mathcal{L}\left\{\frac{d^4y}{dt^4}\right\} &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)
\end{aligned}$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y^{(4)} - 4y\} = \mathcal{L}\{0\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y^{(4)}\} - 4\mathcal{L}\{y\} = 0$$

$$[s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] - 4Y(s) = 0$$

Plug in the initial conditions, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$, and $y'''(0) = 0$.

$$[s^4Y(s) - s^3 + 2s] - 4Y(s) = 0$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$s^4Y(s) - 4Y(s) - s^3 + 2s = 0$$

$$(s^4 - 4)Y(s) = s^3 - 2s$$

$$\begin{aligned}
Y(s) &= \frac{s^3 - 2s}{s^4 - 4} \\
&= \frac{s(s^2 - 2)}{(s^2 - 2)(s^2 + 2)} \\
&= \frac{s}{s^2 + 2}
\end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} \\ &= \cos\sqrt{2}t \end{aligned}$$