Problem 26

In each of Problems 24 through 27, find the Laplace transform $Y(s) = \mathcal{L}\{y\}$ of the solution of the given initial value problem. A method of determining the inverse transform is developed in Section 6.3. You may wish to refer to Problems 21 through 24 in Section 6.1.

$$y'' + 4y = \begin{cases} t, & 0 \le t < 1, \\ 1, & 1 \le t < \infty; \end{cases} \qquad y(0) = 0, \quad y'(0) = 0$$

Solution

Let $f(t)$ represent the piecewise function on the right side.

$$y'' + 4y = f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < \infty \end{cases}$$

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\, dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{f(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$, and $f(t)$.

$$[s^2Y(s)] + 4Y(s) = \int_0^1 e^{-st}(t)\, dt + \int_1^\infty e^{-st}(1)\, dt$$

$$\begin{aligned} (s^2 + 4)Y(s) &= \int_0^1 te^{-st}\, dt + \int_1^\infty e^{-st}\, dt \\ &= \frac{1 - (s+1)e^{-s}}{s^2} + \frac{e^{-s}}{s} \\ &= \frac{1}{s^2} - \cancel{\frac{e^{-s}}{s}} - \frac{e^{-s}}{s^2} + \cancel{\frac{e^{-s}}{s}} \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} \end{aligned}$$

Divide both sides by $s^2 + 4$.

$$\begin{aligned} Y(s) &= \frac{1}{s^2(s^2+4)} - \frac{e^{-s}}{s^2(s^2+4)} \\ &= \frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4} - \left(\frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}\right)e^{-s} \end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$. Note that $H(t)$ is the Heaviside function, which is defined to be 1 if $t > 0$ and 0 if $t < 0$.

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4} - \left(\frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}\right)e^{-s}\right\} \\ &= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{\left(\frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}\right)e^{-s}\right\} \\ &= \frac{1}{4}t - \frac{1}{8}\sin 2t - \left\{\frac{1}{4}(t-1) - \frac{1}{8}\sin[2(t-1)]\right\}H(t-1) \end{aligned}$$