Problem 36

Consider Bessel's equation of order zero

$$ty'' + y' + ty = 0.$$

Recall from Section 5.7 that $t = 0$ is a regular singular point for this equation, and therefore solutions may become unbounded as $t \to 0$. However, let us try to determine whether there are any solutions that remain finite at $t = 0$ and have finite derivatives there. Assuming that there is such a solution $y = \phi(t)$, let $Y(s) = \mathcal{L}\{\phi(t)\}$.

(a) Show that $Y(s)$ satisfies

$$(1 + s^2)Y'(s) + sY(s) = 0.$$

(b) Show that $Y(s) = c(1 + s^2)^{-1/2}$, where c is an arbitrary constant.

(c) Writing $(1 + s^2)^{-1/2} = s^{-1}(1 + s^{-2})^{-1/2}$, expanding in a binomial series valid for $s > 1$, and assuming that it is permissible to take the inverse transform term by term, show that

$$y = c\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = cJ_0(t),$$

where J_0 is the Bessel function of the first kind of order zero. Note that $J_0(0) = 1$ and that J_0 has finite derivatives of all orders at $t = 0$. It was shown in Section 5.7 that the second solution of this equation becomes unbounded as $t \to 0$.