Problem 39

In this problem we show how a general partial fraction expansion can be used to calculate many inverse Laplace transforms. Suppose that

$$F(s) = P(s)/Q(s),$$

where $Q(s)$ is a polynomial of degree n with distinct zeros $r_1, \ldots, r_n$, and $P(s)$ is a polynomial of degree less than n. In this case it is possible to show that $P(s)/Q(s)$ has a partial fraction expansion of the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \cdots + \frac{A_n}{s - r_n}, \tag{i}$$

where the coefficients $A_1, \ldots, A_n$ must be determined.

(a) Show that

$$A_k = P(r_k)/Q'(r_k), \qquad k = 1, \ldots, n. \tag{ii}$$

Hint: One way to do this is to multiply Eq. (i) by $s - r_k$ and then to take the limit as $s \to r_k$.

(b) Show that

$$\mathcal{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \frac{P(r_k)}{Q'(r_k)} e^{r_k t}. \tag{iii}$$

Solution

Part (a)

Multiply both sides of Eq. (i) by $s - r_k$, where $k = 0, 1, \ldots, n$.

$$\frac{P(s)}{Q(s)}(s - r_k) = \frac{A_1}{s - r_1}(s - r_k) + \cdots + A_k + \cdots + \frac{A_n}{s - r_n}(s - r_k)$$

Take the limit of both sides as $s \to r_k$. All terms on the right side vanish as a result except for A_k.

$$\begin{aligned}
\lim_{s \to r_k} \frac{P(s)}{Q(s)}(s - r_k) &= \lim_{s \to r_k} \left[\frac{A_1}{s - r_1}(s - r_k) + \cdots + A_k + \cdots + \frac{A_n}{s - r_n}(s - r_k)\right] \\
&= \lim_{s \to r_k} \frac{A_1}{s - r_1}(s - r_k) + \cdots + \lim_{s \to r_k} A_k + \cdots + \lim_{s \to r_k} \frac{A_n}{s - r_n}(s - r_k) \\
&= A_k
\end{aligned}$$

Use l'Hôpital's rule for the remaining limit since $Q(r_k) = 0$.

$$\begin{aligned}
A_k &= \lim_{s \to r_k} \frac{P(s)}{Q(s)}(s - r_k) \\
&\overset{\frac{0}{0}}{\underset{\text{H}}{=}} \lim_{s \to r_k} \frac{P'(s)(s - r_k) + P(s)}{Q'(s)} \\
&= \frac{P(r_k)}{Q'(r_k)}
\end{aligned}$$

Part (b)

Take the inverse Laplace transform of $F(s)$.

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{A_1}{s-r_1} + \cdots + \frac{A_n}{s-r_n}\right\} \\
&= A_1\mathcal{L}^{-1}\left\{\frac{1}{s-r_1}\right\} + \cdots A_n\mathcal{L}^{-1}\left\{\frac{1}{s-r_n}\right\} \\
&= \frac{P(r_1)}{Q'(r_1)}\mathcal{L}^{-1}\left\{\frac{1}{s-r_1}\right\} + \cdots \frac{P(r_n)}{Q'(r_n)}\mathcal{L}^{-1}\left\{\frac{1}{s-r_n}\right\} \\
&= \frac{P(r_1)}{Q'(r_1)}e^{r_1 t} + \cdots + \frac{P(r_n)}{Q'(r_n)}e^{r_n t} \\
&= \sum_{k=1}^{n} \frac{P(r_k)}{Q'(r_k)}e^{r_k t}
\end{aligned}$$