Problem 13

In each of Problems 13 through 18, find the Laplace transform of the given function.

$$f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \geq 2 \end{cases}$$

Solution

The Laplace transform of a function $f(t)$ is defined here as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

For the provided function in particular, we have

$$\begin{aligned}
F(s) &= \int_0^2 e^{-st}(0)\, dt + \int_2^\infty e^{-st}(t-2)^2\, dt \\
&= \int_2^\infty e^{-st}(t^2 - 4t + 4)\, dt \\
&= \int_2^\infty t^2 e^{-st}\, dt - 4\int_2^\infty t e^{-st}\, dt + 4\int_2^\infty e^{-st}\, dt \\
&= \int_2^\infty \frac{\partial^2}{\partial s^2}(e^{-st})\, dt + 4\int_2^\infty \frac{\partial}{\partial s}(e^{-st})\, dt + 4\int_2^\infty e^{-st}\, dt \\
&= \frac{d^2}{ds^2}\int_2^\infty e^{-st}\, dt + 4\frac{d}{ds}\int_2^\infty e^{-st}\, dt + 4\int_2^\infty e^{-st}\, dt \\
&= \frac{d^2}{ds^2}\left(-\frac{1}{s}e^{-st}\right)\bigg|_2^\infty + 4\frac{d}{ds}\left(-\frac{1}{s}e^{-st}\right)\bigg|_2^\infty + 4\left(-\frac{1}{s}e^{-st}\right)\bigg|_2^\infty \\
&= \frac{d^2}{ds^2}\left(\frac{1}{s}e^{-2s}\right) + 4\frac{d}{ds}\left(\frac{1}{s}e^{-2s}\right) + 4\left(\frac{1}{s}e^{-2s}\right) \\
&= \frac{d}{ds}\left(\frac{-2se^{-2s} - e^{-2s}}{s^2}\right) + 4\left(\frac{-2se^{-2s} - e^{-2s}}{s^2}\right) + 4\left(\frac{1}{s}e^{-2s}\right) \\
&= \frac{(-2e^{-2s} + 4se^{-2s} + 2e^{-2s})s^2 - 2s(-2se^{-2s} - e^{-2s})}{s^4} + 4\left(\frac{-2se^{-2s} - e^{-2s}}{s^2}\right) + 4\left(\frac{1}{s}e^{-2s}\right) \\
&= \frac{2se^{-2s} + 4s^2e^{-2s} + 4s^3e^{-2s}}{s^4} + \frac{-8se^{-2s} - 4e^{-2s}}{s^2} + \frac{4e^{-2s}}{s} \\
&= \frac{2e^{-2s} + 4se^{-2s} + 4s^2e^{-2s}}{s^3} + \frac{-8s^2e^{-2s} - 4se^{-2s}}{s^3} + \frac{4s^2e^{-2s}}{s^3} \\
&= \frac{2e^{-2s}}{s^3}.
\end{aligned}$$

This answer is in disagreement with the one at the back of the book, $F(s) = 2e^{-s}/s^3$.