Problem 14

In each of Problems 13 through 18, find the Laplace transform of the given function.

$$f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \geq 1 \end{cases}$$

Solution

The Laplace transform of a function $f(t)$ is defined here as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

For the provided function in particular, we have

$$\begin{aligned}
F(s) &= \int_0^1 e^{-st}(0)\, dt + \int_1^\infty e^{-st}(t^2 - 2t + 2)\, dt \\
&= \int_1^\infty t^2 e^{-st}\, dt - 2\int_1^\infty t e^{-st}\, dt + 2\int_1^\infty e^{-st}\, dt \\
&= \int_1^\infty \frac{\partial^2}{\partial s^2}(e^{-st})\, dt + 2\int_1^\infty \frac{\partial}{\partial s}(e^{-st})\, dt + 2\int_1^\infty e^{-st}\, dt \\
&= \frac{d^2}{ds^2}\int_1^\infty e^{-st}\, dt + 2\frac{d}{ds}\int_1^\infty e^{-st}\, dt + 2\int_1^\infty e^{-st}\, dt \\
&= \frac{d^2}{ds^2}\left(-\frac{1}{s}e^{-st}\right)\bigg|_1^\infty + 2\frac{d}{ds}\left(-\frac{1}{s}e^{-st}\right)\bigg|_1^\infty + 2\left(-\frac{1}{s}e^{-st}\right)\bigg|_1^\infty \\
&= \frac{d^2}{ds^2}\left(\frac{1}{s}e^{-s}\right) + 2\frac{d}{ds}\left(\frac{1}{s}e^{-s}\right) + 2\left(\frac{1}{s}e^{-s}\right) \\
&= \frac{d}{ds}\left(-\frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right) + 2\left(-\frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right) + 2\left(\frac{1}{s}e^{-s}\right) \\
&= \left(\frac{2}{s^3}e^{-s} + \frac{1}{s^2}e^{-s} + \frac{1}{s^2}e^{-s} + \frac{1}{s}e^{-s}\right) + 2\left(-\frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right) + 2\left(\frac{1}{s}e^{-s}\right) \\
&= \frac{2 + s + s + s^2 - 2s - 2s^2 + 2s^2}{s^3}e^{-s} \\
&= \frac{2 + s^2}{s^3}e^{-s}.
\end{aligned}$$