Problem 32

In each of Problems 30 through 33, find the Laplace transform of the given function. In Problem 33, assume that term-by-term integration of the infinite series is permissible.

$$f(t) = 1 - u_1(t) + \cdots + u_{2n}(t) - u_{2n+1}(t) = 1 + \sum_{k=1}^{2n+1} (-1)^k u_k(t)$$

Solution

The Laplace transform of a function $f(t)$ is defined to be

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Of course, for this integral to converge, it's necessary that $s > 0$. Substitute the given function for $f(t)$, replacing $u_k(t)$ with the Heaviside function $H(t-k)$.

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} \left[1 + \sum_{k=1}^{2n+1} (-1)^k H(t-k)\right] dt \\
&= \int_0^\infty e^{-st}\, dt + \int_0^\infty e^{-st} \sum_{k=1}^{2n+1} (-1)^k H(t-k)\, dt \\
&= \int_0^\infty e^{-st}\, dt + \sum_{k=1}^{2n+1} (-1)^k \int_0^\infty e^{-st} H(t-k)\, dt \\
&= \int_0^\infty e^{-st}\, dt + \sum_{k=1}^{2n+1} (-1)^k \int_k^\infty e^{-st}\, dt \\
&= \left(-\frac{1}{s} e^{-st}\right)\Big|_0^\infty + \sum_{k=1}^{2n+1} (-1)^k \left(-\frac{1}{s} e^{-st}\right)\Big|_k^\infty \\
&= \frac{1}{s} + \sum_{k=1}^{2n+1} (-1)^k \left(\frac{1}{s} e^{-ks}\right) \\
&= \frac{1}{s} + \frac{1}{s} \sum_{k=1}^{2n+1} (-1)^k e^{-ks} \\
&= \frac{1}{s}\left[1 + \sum_{k=1}^{2n+1} (-1)^k e^{-ks}\right] \\
&= \frac{1}{s}\left[1 + \sum_{k=1}^{2n+1} (-e^{-s})^k\right] \\
&= \frac{1}{s} \sum_{k=0}^{2n+1} (-e^{-s})^k \\
&= \frac{1}{s}\left[\frac{-1 + (-e^{-s})^{(2n+1)+1}}{-1 + (-e^{-s})}\right] = \frac{1 - e^{-2s(n+1)}}{s(1 + e^{-s})}
\end{aligned}$$