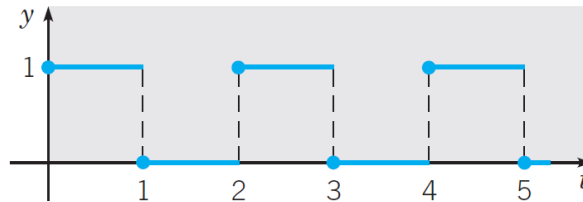


### Problem 33

In each of Problems 30 through 33, find the Laplace transform of the given function. In Problem 33, assume that term-by-term integration of the infinite series is permissible.

$$f(t) = 1 + \sum_{k=1}^{\infty} (-1)^k u_k(t). \quad \text{See Figure 6.3.7.}$$



**FIGURE 6.3.7** The function  $f(t)$  in Problem 33; a square wave.

#### Solution

The Laplace transform of a function  $f(t)$  is defined to be

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

Of course, for this integral to converge, it's necessary that  $s > 0$ . Substitute the given function for  $f(t)$ , replacing  $u_k(t)$  with the Heaviside function  $H(t - k)$ .

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} \left[ 1 + \sum_{k=1}^{\infty} (-1)^k H(t - k) \right] dt \\ &= \int_0^{\infty} e^{-st} dt + \int_0^{\infty} e^{-st} \sum_{k=1}^{\infty} (-1)^k H(t - k) dt \\ &= \int_0^{\infty} e^{-st} dt + \sum_{k=1}^{\infty} (-1)^k \int_0^{\infty} e^{-st} H(t - k) dt \\ &= \int_0^{\infty} e^{-st} dt + \sum_{k=1}^{\infty} (-1)^k \int_k^{\infty} e^{-st} dt \\ &= \left( -\frac{1}{s} e^{-st} \right) \Big|_0^{\infty} + \sum_{k=1}^{\infty} (-1)^k \left( -\frac{1}{s} e^{-st} \right) \Big|_k^{\infty} \\ &= \frac{1}{s} + \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{s} e^{-ks} \right) \\ &= \frac{1}{s} + \frac{1}{s} \sum_{k=1}^{\infty} (-1)^k e^{-ks} \\ &= \frac{1}{s} \left[ 1 + \sum_{k=1}^{\infty} (-e^{-s})^k \right] = \frac{1}{s} \sum_{k=0}^{\infty} (-e^{-s})^k = \frac{1}{s} \left[ \frac{1}{1 - (-e^{-s})} \right] = \frac{1}{s(1 + e^{-s})} \end{aligned}$$