

Problem 35

In each of Problems 35 through 38, use the result of Problem 34 to find the Laplace transform of the given function.

$$f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & 1 \leq t < 2; \end{cases}$$

$$f(t+2) = f(t).$$

Compare with Problem 33.

Solution

For a function that repeats itself periodically every T units, the Laplace transform is

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-sT}}.$$

In this problem the period is $T = 2$. Therefore,

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \frac{\int_0^2 e^{-st} f(t) dt}{1 - e^{-2s}} \\ &= \frac{\int_0^1 e^{-st}(1) dt + \int_1^2 e^{-st}(0) dt}{1 - e^{-2s}} \\ &= \frac{\int_0^1 e^{-st} dt}{1 - e^{-2s}} \\ &= \frac{\left(-\frac{1}{s}e^{-st}\right)\Big|_0^1}{1 - e^{-2s}} \\ &= \frac{\frac{1}{s} - \frac{1}{s}e^{-s}}{1 - e^{-2s}} \\ &= \frac{1 - e^{-s}}{s(1 - e^{-2s})} \\ &= \frac{1 - e^{-s}}{s(1 + e^{-s})(1 - e^{-s})} \\ &= \frac{1}{s(1 + e^{-s})}. \end{aligned}$$