Problem 37

In each of Problems 35 through 38, use the result of Problem 34 to find the Laplace transform of the given function.

$$f(t) = t, \qquad 0 \le t < 1;$$
$$f(t+1) = f(t).$$
See Figure 6.3.9.

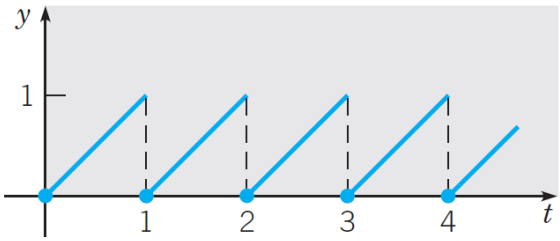

FIGURE 6.3.9 The function $f(t)$ in Problem 37; a sawtooth wave.

Solution

For a function that repeats itself periodically every T units, the Laplace transform is

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}.$$

In this problem the period is $T = 1$. Therefore,

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \frac{\displaystyle\int_0^1 e^{-st}(t)\,dt}{1 - e^{-s}} \\
&= \frac{\displaystyle\int_0^1 \left(-\frac{\partial}{\partial s} e^{-st}\right) dt}{1 - e^{-s}} \\
&= \frac{-\frac{d}{ds}\int_0^1 e^{-st}\,dt}{1 - e^{-s}} \\
&= \frac{-\frac{d}{ds}\left(-\frac{1}{s} e^{-st}\Big|_0^1\right)}{1 - e^{-s}} \\
&= \frac{-\frac{d}{ds}\left(\frac{1}{s} - \frac{1}{s} e^{-s}\right)}{1 - e^{-s}} \\
&= \frac{-\left(-\frac{1}{s^2} + \frac{1}{s^2} e^{-s} + \frac{1}{s} e^{-s}\right)}{1 - e^{-s}} \\
&= \frac{1 - (1+s)e^{-s}}{s^2(1 - e^{-s})}.
\end{aligned}$$