Problem 39

(a) If $f(t) = 1 - u_1(t)$, find $\mathcal{L}\{f(t)\}$; compare with Problem 30. Sketch the graph of $y = f(t)$.

(b) Let $g(t) = \int_0^t f(\xi)\,d\xi$, where the function f is defined in part (a). Sketch the graph of $y = g(t)$ and find $\mathcal{L}\{g(t)\}$.

(c) Let $h(t) = g(t) - u_1(t)g(t-1)$, where g is defined in part (b). Sketch the graph of $y = h(t)$ and find $\mathcal{L}\{h(t)\}$.

Solution

The Laplace transform of a function $f(t)$ is defined to be

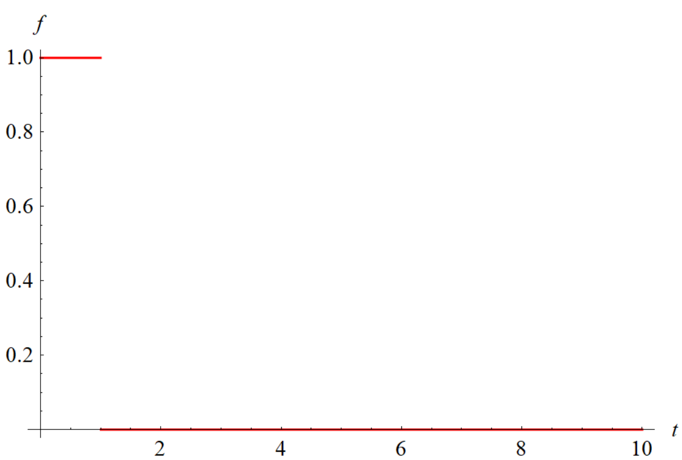
$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

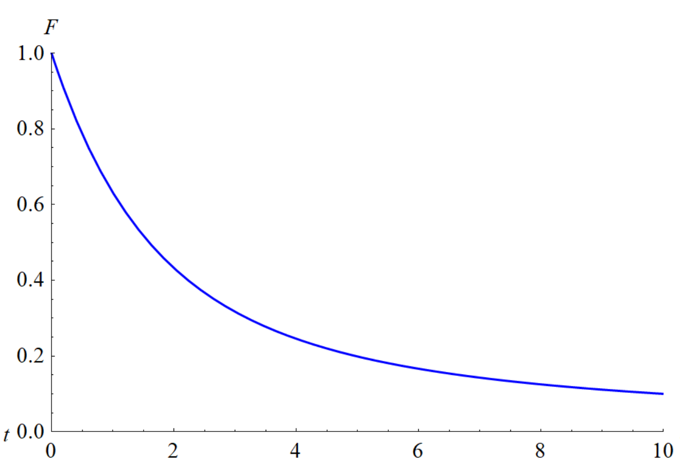
Part (a)

Substitute the given function for $f(t)$, using the Heaviside function $H(t-1)$ for $u_1(t)$.

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st}[1 - H(t-1)]\,dt \\
&= \int_0^\infty e^{-st}\,dt - \int_0^\infty e^{-st} H(t-1)\,dt \\
&= \int_0^\infty e^{-st}\,dt - \int_1^\infty e^{-st}\,dt \\
&= \left(-\frac{1}{s}e^{-st}\right)\Bigg|_0^\infty - \left(-\frac{1}{s}e^{-st}\right)\Bigg|_1^\infty \\
&= \frac{1}{s} - \frac{1}{s}e^{-s} \\
&= \frac{1 - e^{-s}}{s}
\end{aligned}$$

Below is a side-by-side comparison of $f(t)$ and $F(s)$.





Part (b)

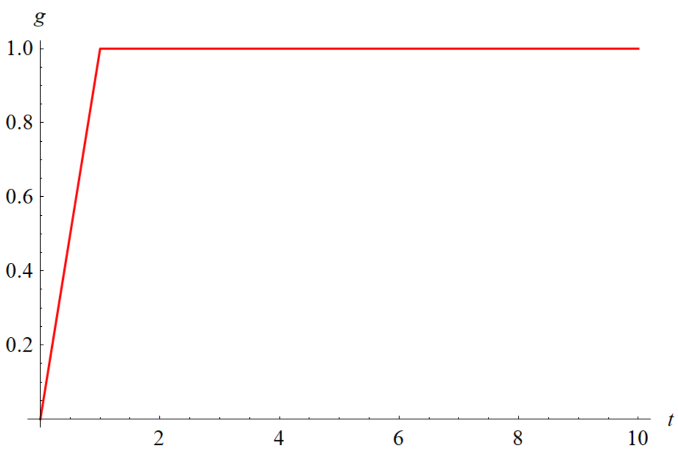
Start by evaluating $g(t)$.

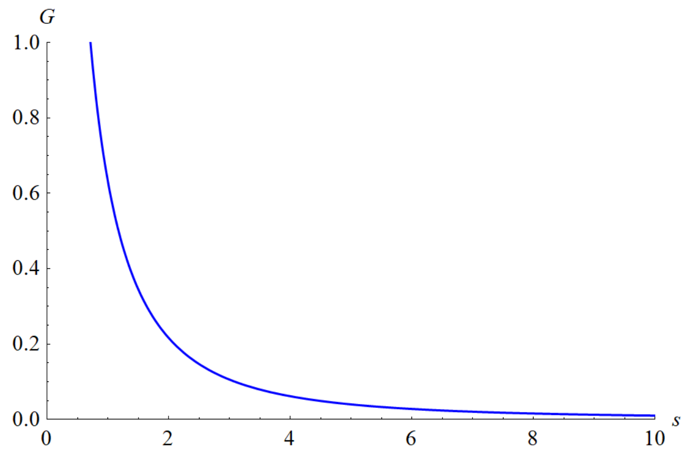
$$\begin{aligned}
g(t) &= \int_0^t f(\xi)\,d\xi \\
&= \int_0^t [1 - H(\xi - 1)]\,d\xi \\
&= \int_0^t d\xi - \int_0^t H(\xi - 1)\,d\xi \\
&= \int_0^t d\xi - H(t-1)\int_1^t d\xi \\
&= t - (t-1)H(t-1)
\end{aligned}$$

Now find the Laplace transform.

$$\begin{aligned}
\mathcal{L}\{g(t)\} &= \int_0^\infty e^{-st} g(t)\,dt \\
&= \int_0^\infty e^{-st}[t - (t-1)H(t-1)]\,dt \\
&= \int_0^\infty te^{-st}\,dt - \int_0^\infty e^{-st}(t-1)H(t-1)\,dt \\
&= \int_0^\infty \left(-\frac{\partial}{\partial s}e^{-st}\right)dt - \int_1^\infty e^{-st}(t-1)\,dt \\
&= -\frac{d}{ds}\int_0^\infty e^{-st}\,dt - \int_1^\infty te^{-st}\,dt + \int_1^\infty e^{-st}\,dt \\
&= -\frac{d}{ds}\left(-\frac{1}{s}e^{-st}\bigg|_0^\infty\right) - \int_1^\infty \left(-\frac{\partial}{\partial s}e^{-st}\right)dt + \left(-\frac{1}{s}e^{-st}\right)\bigg|_1^\infty \\
&= -\frac{d}{ds}\left(\frac{1}{s}\right) + \frac{d}{ds}\int_1^\infty e^{-st}\,dt + \left(\frac{1}{s}e^{-s}\right) \\
&= \frac{1}{s^2} + \frac{d}{ds}\left(-\frac{1}{s}e^{-st}\bigg|_1^\infty\right) + \frac{1}{s}e^{-s} \\
&= \frac{1 + se^{-s}}{s^2} + \frac{d}{ds}\left(\frac{1}{s}e^{-s}\right) \\
&= \frac{1 + se^{-s}}{s^2} + \left(-\frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right) \\
&= \frac{1 + \cancel{se^{-s}} - e^{-s} - \cancel{se^{-s}}}{s^2} \\
&= \frac{1 - e^{-s}}{s^2}
\end{aligned}$$

Below is a side-by-side comparison of $g(t)$ and $G(s)$.





Part (c)

Start by finding what $h(t)$ is.

$$\begin{aligned} h(t) &= g(t) - u_1(t)g(t-1) \\ &= [t - (t-1)H(t-1)] - H(t-1)\{(t-1) - [(t-1)-1]H[(t-1)-1]\} \\ &= t - (t-1)H(t-1) - H(t-1)[t-1-(t-2)H(t-2)] \\ &= t - (t-1)H(t-1) - (t-1)H(t-1) + (t-2)H(t-2)H(t-1) \\ &= t - 2(t-1)H(t-1) + (t-2)H(t-2) \end{aligned}$$

Now find the Laplace transform.

$$\begin{aligned} \mathcal{L}\{h(t)\} &= \int_0^\infty e^{-st}h(t)\,dt \\ &= \int_0^\infty e^{-st}[t - 2(t-1)H(t-1) + (t-2)H(t-2)]\,dt \\ &= \int_0^\infty te^{-st}\,dt - 2\int_0^\infty e^{-st}(t-1)H(t-1)\,dt + \int_0^\infty e^{-st}(t-2)H(t-2)\,dt \\ &= \int_0^\infty te^{-st}\,dt - 2\int_0^\infty e^{-st}(t-1)H(t-1)\,dt + \int_0^\infty e^{-st}(t-2)H(t-2)\,dt \\ &= \int_0^\infty te^{-st}\,dt - 2\int_1^\infty e^{-st}(t-1)\,dt + \int_2^\infty e^{-st}(t-2)\,dt \\ &= \int_0^\infty te^{-st}\,dt - 2\int_1^\infty te^{-st}\,dt + 2\int_1^\infty e^{-st}\,dt + \int_2^\infty te^{-st}\,dt - 2\int_2^\infty e^{-st}\,dt \\ &= \int_0^\infty \left(-\frac{\partial}{\partial s}e^{-st}\right)dt - 2\int_1^\infty \left(-\frac{\partial}{\partial s}e^{-st}\right)dt + 2\int_1^\infty e^{-st}\,dt + \int_2^\infty \left(-\frac{\partial}{\partial s}e^{-st}\right)dt + 2\int_\infty^2 e^{-st}\,dt \\ &= -\frac{d}{ds}\int_0^\infty e^{-st}\,dt + 2\frac{d}{ds}\int_1^\infty e^{-st}\,dt - \frac{d}{ds}\int_2^\infty e^{-st}\,dt + 2\int_1^2 e^{-st}\,dt \end{aligned}$$

Evaluate the integrals and simplify the result.

$$\begin{aligned}
h(t) &= -\frac{d}{ds}\left(-\frac{1}{s}e^{-st}\bigg|_0^{\infty}\right) + 2\frac{d}{ds}\left(-\frac{1}{s}e^{-st}\bigg|_1^{\infty}\right) - \frac{d}{ds}\left(-\frac{1}{s}e^{-st}\bigg|_2^{\infty}\right) + 2\left(-\frac{1}{s}e^{-st}\bigg|_1^{2}\right) \\
&= -\frac{d}{ds}\left(\frac{1}{s}\right) + 2\frac{d}{ds}\left(\frac{1}{s}e^{-s}\right) - \frac{d}{ds}\left(\frac{1}{s}e^{-2s}\right) + 2\left(\frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}\right) \\
&= \frac{1}{s^2} + 2\left(-\frac{1}{s^2}e^{-s} - \cancel{\frac{1}{s}e^{-s}}\right) - \left(-\frac{1}{s^2}e^{-2s} - \cancel{\frac{2}{s}e^{-2s}}\right) + 2\left(\cancel{\frac{1}{s}e^{-s}} - \cancel{\frac{1}{s}e^{-2s}}\right) \\
&= \frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} \\
&= \frac{1 - 2e^{-s} + e^{-2s}}{s^2} \\
&= \frac{(1 - e^{-s})^2}{s^2} \\
&= \left(\frac{1 - e^{-s}}{s}\right)^2
\end{aligned}$$

Below is a side-by-side comparison of $h(t)$ and $H(s)$.